Math 132, Test 1, February 5, 2009 Name: ______________________________.

No graphing calculators or electronic communication of any kind. If you need extra paper please ask. Credit will be awarded for correct content and clarity of presentation. Prepare for math battle. This test has 100 points. There is a take-home bonus problem at the end with instructions. Make sure to at least attempt each part except the bonus.

1. [12pts.] Circle true or false (no partial credit will be awarded in this problem)

a.) TRUE or FALSE: If $f(-x) = -f(x)$ for all $x \in \mathbb{R}$ then the function f is said to be an odd function and one can show that $\int_{-t}^{t} f(x)dx = 0$, for any $t \in \mathbb{R}$.

$$\int_{-t}^{t} f(x)dx = \int_{-t}^{0} f(x)dx + \int_{0}^{t} f(x)dx = \int_{t}^{0} f(-u)(-du) + \int_{0}^{t} f(x)dx = 0.$$

b.) TRUE or FALSE: If $f(-x) = -f(x)$ for all $x \in \mathbb{R}$ then the function f is said to be an odd function and one can show that $\int_{-\infty}^{\infty} f(x)dx = 0$.

Consider $f(x) = \frac{1}{x}$.

c.) TRUE or FALSE: there are real numbers A, B such that

$$\frac{x^2}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

Need to do long-division first.

d.) TRUE or FALSE: there are real numbers A, B, C, D such that

$$\frac{x^2}{(x+2)^2(x^2+4)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cx+D}{x^2+4}$$

e.) TRUE or FALSE: For all $x \in \mathbb{R}$, $\sin(2x) = 2\sin(x)\cos(x)$.

f.) TRUE or FALSE: For all $A, B \in \mathbb{R}$, $\sin(A+B) = \sin(A)\cos(B) - \cos(B)\sin(A)$.

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

2. [20pts.] Calculate the integral below. Your answer should only have one arbitrary constant of integration.

$$\int \frac{6}{x^2+5x+6}\,dx$$

$$\frac{6}{x^2+5x+6} = \frac{6}{(x+3)(x+2)} = \frac{A}{x+3} + \frac{B}{x+2}$$

$$6 = A(x+2) + B(x+3)$$

$$\underline{x=-2} \quad 6 = B$$

$$\underline{x=-3} \quad 6 = -A \Rightarrow A = -6$$

Thus,

$$\int \frac{6}{x^2+5x+6}\,dx = \int\left(\frac{-6}{x+3} + \frac{6}{x+2}\right)dx$$

$$= \boxed{-6\ln|x+3| + 6\ln|x+2| + C}$$

3.[13pts] Use the imaginary exponential technique to determine a correct set of values for the constants A, B, C, D in the following identity:

$$\cos(3\theta)\sin(7\theta) = A\sin(B\theta) + C\sin(D\theta)$$

$$\cos 3\theta \sin 7\theta = \frac{1}{2}\left(e^{3i\theta} + e^{-3i\theta}\right)\frac{1}{2i}\left(e^{7i\theta} - e^{-7i\theta}\right)$$

$$= \frac{1}{4i}\left(e^{10i\theta} - e^{-10i\theta} + e^{4i\theta} - e^{-4i\theta}\right)$$

$$= \frac{1}{2}\frac{1}{2i}\left(e^{10i\theta} - e^{-10i\theta}\right) + \frac{1}{2}\frac{1}{2i}\left(e^{4i\theta} - e^{-4i\theta}\right)$$

$$= \frac{1}{2}\sin(10\theta) + \frac{1}{2}\sin(4\theta)$$

$$= A\sin(B\theta) + C\sin(D\theta).$$

Comparing yields

$A = \frac{1}{2}$, $B = 10$, $C = \frac{1}{2}$, $D = 4$

(OR $A = \frac{1}{2}$, $B = 4$, $C = \frac{1}{2}$, $D = 10$)

Use the identity you derived to integrate the following:

$$\int \cos(3\theta)\sin(7\theta)\,d\theta = \int\left[\frac{1}{2}\sin(10\theta) + \frac{1}{2}\sin(4\theta)\right]d\theta$$

$$= \boxed{\frac{-1}{20}\cos(10\theta) - \frac{1}{8}\cos(4\theta) + C}$$

4. [20pts.] Use a trigonometric substitution to solve the following integral:

$$\int \sqrt{4 - x^2}\, dx$$

Let $x = 2\sin\theta$ then $dx = 2\cos\theta\, d\theta$ and

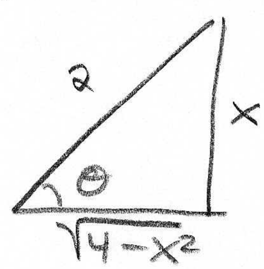
$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4(1-\sin^2\theta)} = \sqrt{4\cos^2\theta} = 2\cos\theta.$

Thus,

$$\begin{aligned}
\int\sqrt{4-x^2}\,dx &= \int(2\cos\theta)(2\cos\theta\, d\theta) \\
&= 4\int \cos^2\theta\, d\theta \\
&= 4\int \tfrac{1}{2}(1+\cos 2\theta)\, d\theta \\
&= 2\left(\theta + \tfrac{1}{2}\sin(2\theta)\right) + C \\
&= 2\theta + \sin(2\theta) + C \\
&= \boxed{2\sin^{-1}\left(\tfrac{x}{2}\right) + \sin\left[2\sin^{-1}\left(\tfrac{x}{2}\right)\right] + C}
\end{aligned}$$

We can simplify the middle term,

$$\begin{aligned}
\sin(2\theta) &= 2\sin\theta\cos\theta \\
&= 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) \\
&= \frac{x\sqrt{4-x^2}}{2}
\end{aligned}$$



$$\boxed{\int\sqrt{4-x^2}\,dx = 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C}$$

5.[20pts] Determine if the following integrals converge or diverge. If the integral converges calculate the value to which it converges. Otherwise, explain why it diverges. *(Part a. is worth 15 pts, 10 of those pts for integrating correctly.)*

a.) $\displaystyle\int_0^\infty xe^{-x}\,dx$

First find the anti derivative,

$$\int \underbrace{x}_{u}\underbrace{e^{-x}dx}_{dV} = uv - \int v\,du$$

$$\boxed{\begin{array}{ll} u = x & du = dx \\ dV = e^{-x}dx & V = -e^{-x} \end{array}}$$

$$= -xe^{-x} + \int e^{-x}dx$$

$$= -xe^{-x} - e^{-x} + C$$

Now apply the def$^{\underline{n}}$ of the improper integral,

$$\int_0^\infty xe^{-x}dx = \lim_{t\to\infty} \int_0^t xe^{-x}dx$$

$$= \lim_{t\to\infty}\left(-xe^{-x}\Big|_0^t - e^{-t}\Big|_0^t\right)$$

$$= \lim_{t\to\infty}\left(-te^{-t} + 0 - \underset{0}{e^{-t}} + e^0\right)$$

$$= \lim_{t\to\infty}\left(\frac{-t}{e^t}\right) + 1 \underset{\left(\frac{\infty}{\infty}\right)}{\overset{\mathcal{L}}{=}} \lim_{t\to\infty}\left(\underset{0}{\frac{-1}{e^t}}\right) + 1 = \boxed{1}$$

b.) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x}}\,dx = \lim_{t\to 1^-}\int_0^t \frac{dx}{\sqrt{1-x}}$

$$= \lim_{t\to 1^-}\left(-2\sqrt{1-x}\,\Big|_0^t\right.$$

$$= \lim_{t\to 1^-}\left(-2\sqrt{1-t} + 2\sqrt{1}\right)$$

$$= \boxed{2}$$

6.[20pts] Calculate $\displaystyle\int \frac{x^2+5x+4}{x^3+3x^2+x-5}\,dx$.

Useful fact: the denominator $x^3+3x^2+x-5=0$ *when* $x=1$.

$$\begin{aligned} x^3+3x^2+x-5 &= (x-1)(x^2+Ax+B) \\ &= x^3+Ax^2+Bx-x^2-Ax-B \\ &= x^3+x^2(A-1)+x(B-A)-B \end{aligned}$$

Comparing coefficients, $B=5$ and $B-A=1 \Rightarrow A=B-1=4$.
Hence, $x^3+3x^2+x-5=(x-1)(x^2+4x+5)$. Consider

$$\frac{x^2+5x+4}{x^3+3x^2+x-5} = \frac{x^2+5x+4}{(x-1)(x^2+4x+5)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4x+5}$$

$$\begin{aligned} \Rightarrow x^2+5x+4 &= A(x^2+4x+5)+(Bx+C)(x-1) \\ &= Ax^2+4Ax+5A+Bx^2-B+Cx-C \\ &= x^2[A+B]+x[4A+C]+5A-B-C \end{aligned}$$

Equate Coefficients,

$x^2)$ $1=A+B$ $\longrightarrow$ $9=9A+9B$

$x)$ $5=4A+C$ $\longrightarrow$ $9=9A-B$

$x^0)$ $4=5A-B-C$

$0=10B \Rightarrow B=0$

Then $1=A+0 \Rightarrow A=1$ then $5=4+C \Rightarrow C=1$.

$$\begin{aligned} \int \frac{x^2+5x+4}{x^3+3x^2+x-5}\,dx &= \int \frac{dx}{x-1} + \int \frac{dx}{x^2+4x+5} \\ &= \ln|x-1| + \int \frac{dx}{(x+2)^2+1} \quad : \text{ let } w=x+2. \\ &= \ln|x-1| + \int \frac{dw}{w^2+1} \\ &= \ln|x-1| + \tan^{-1}(w)+C \\ &= \boxed{\ln|x-1| + \tan^{-1}(x+2)+C} \end{aligned}$$