Math 132, Test 2, March 3, 2009 Name: ______________________.

No graphing calculators or electronic communication of any kind. If you need extra paper please ask. Credit will be awarded for correct content and clarity of presentation. Prepare for math battle. This test has 100 points. There is a take-home bonus problem at the end with instructions. Make sure to at least attempt each part except the bonus.

1. [20pts.] Solve the differential equation below.

$$\frac{dy}{dx} = \frac{1}{(x^2-5x-6)\sqrt{4-y^2}} \Rightarrow \sqrt{4-y^2}\,dy = \frac{dx}{x^2-5x-6}$$

Let's attach these integrals one at a time,

$$\frac{1}{x^2-5x-6} = \frac{1}{(x-6)(x+1)} = \frac{A}{x-6} + \frac{B}{x+1}$$

$$1 = A(x+1) + B(x-6)$$

$\underline{x=-1)}$ $1 = -7B$ $\therefore$ $B = -1/7$

$\underline{x=6)}$ $1 = 7A$ $\therefore$ $A = 1/7$

$$\int \frac{dx}{x^2-5x-6} = \frac{1}{7}\int\left(\frac{1}{x-6} - \frac{1}{x+1}\right)dx = \underline{\frac{1}{7}\ln|x-6| - \ln|x+1| + C}$$

$$\int\sqrt{4-y^2}\,dy = \int\left(\sqrt{4\cos^2\theta}\right)2\cos\theta\,d\theta$$

$y = 2\sin\theta$

$y^2 = 4\sin^2\theta$

$4-y^2 = 4(1-\sin^2\theta) = 4\cos^2\theta$

$dy = 2\cos\theta\,d\theta$

$$= 4\int\cos^2\theta\,d\theta$$

$$= 2\int(1+\cos(2\theta))\,d\theta$$

$$= 2\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C$$

$$= 2(\sin^{-1}(y/2) + \tfrac{1}{2}\sin(2\sin^{-1}(y/2)) + C$$

Thus,

$$\int\sqrt{4-y^2}\,dy = \int\frac{dx}{x^2-5x-6}$$

$$\boxed{2\sin^{-1}(y/2) + \frac{1}{2}\sin(2\sin^{-1}(y/2)) = \frac{1}{7}\left(\ln|x-6| - \ln|x+1|\right) + C}$$

2. [15pts] Solve the differential equation below.

$$\star \quad \boxed{\frac{dy}{dx} + y = e^{-x}} \qquad \text{observe} \quad P(x) = 1 \quad \& \quad Q(x) = e^{-x}.$$

$$\mu = \exp\left(\int 1\,dx\right) = \exp(x) = e^{x}$$

Multiply $\star$ by μ,

$$e^{x}\frac{dy}{dx} + e^{x}y = e^{x}e^{-x} = 1$$

$$\Rightarrow \frac{d}{dx}\left(e^{x}y\right) = 1$$

Integrate both sides w.r.t. x, use FTC on LHS,

$$e^{x}y = \int 1\,dx = x + c$$

$$\therefore \boxed{y = xe^{-x} + ce^{-x}}$$

3. [15pts] Solve the differential equations below.

(a.) $y'' + 2y' + y = 0$

$\lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0 \quad \therefore \lambda_1 = \lambda_2 = -1$

By Recipe, $\boxed{y = c_1 e^{-x} + c_2 x e^{-x}}$

(b.) $y'' + 4y' + 5y = 0$

$\lambda^2 + 4\lambda + 5 = 0 \longrightarrow \lambda = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

Hence $\alpha = -2$ and $\beta = 1$. Use Recipe,

$\boxed{y = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin(x)}$

Or $\underline{y = Ae^{-2x} \cos(x + \phi)}$

(c.) $y'' - y = 0$

$\lambda^2 - 1 = 0 \Rightarrow \lambda^2 = 1 \Rightarrow \underline{\lambda = \pm 1}$

Hence by Recipe,

$\boxed{y = c_1 e^{x} + c_2 e^{-x}}$

Alternatively,

$\underline{y = A\cosh(x) + B\sinh(x)}$

4.[42pts] Determine if the series below converge or diverge. If possible find the value to which they converge. Explain your answer. Make sure to justify any tests which are used. Most of the credit will be awarded on the basis of the clarity and completeness of your answer.

(a.) $\sum_{n=1}^{\infty} \frac{n}{n-2}$ observe $a_n = \frac{n}{n-2} \longrightarrow 1$ as $n \longrightarrow \infty$. Thus $\lim_{n \to \infty} a_n \neq 0$ so by the n^{th} term test $\sum_{n=1}^{\infty} \frac{n}{n-2}$ diverges.

(b.) $\sum_{n=4}^{\infty} \frac{1}{n-3} = \frac{1}{4-3} + \frac{1}{5-3} + \frac{1}{6-3} + \cdots$

$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

$= \sum_{k=1}^{\infty} \frac{1}{n}$ this diverges by $p=1$ Series test. (This is the harmonic series) (c.) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+2}}{4^n} = \frac{3^3}{4} + \frac{3^4}{4^2} + \frac{3^5}{4^3} + \cdots = \sum_{n=1}^{\infty} \frac{3^3}{4}\left(\frac{3}{4}\right)^{n-1}$

identify that this is a geometric series with $a = 27/4$ and $r = 3/4$. Hence this series converges to the value

$$\frac{a}{1-r} = \frac{27/4}{1-3/4} = \frac{27/4}{1/4} = \boxed{27 = \sum_{n=1}^{\infty} \frac{3^{n+2}}{4^n}}$$

(d.) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ this is a $p = 2$ series hence it converges by p-series test.

(e.) [redacted] $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$

Observe $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by $p=2$ series test, hence, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$ converges. Moreover, we can calculate

$$\frac{1}{n^2+3n+2} = \frac{A}{n+1} + \frac{B}{n+2}$$

$$1 = A(n+2) + B(n+1)$$

$$\left.\begin{array}{ll} \underline{n=-2} & 1 = -B \\ \underline{n=-1} & 1 = A \end{array}\right\} \rightarrow \frac{1}{n^2+3n+2} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$S_n = \sum_{k=1}^{n} \left(\frac{1}{k+1} - \frac{1}{k+2}\right) = \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}}\right) + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right).$$

[redacted] Thus we find a nice formula for S_n and

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} = \lim_{n\to\infty}\left(\frac{1}{2} - \underbrace{\frac{1}{n+2}}_{0}\right) = \boxed{\frac{1}{2}}.$$

(f.) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n!)^2}$ use ratio test.

$$L = \lim_{n\to\infty} \left| \frac{(n!)^2}{[(n+1)!]^2} \right| = \lim_{n\to\infty} \left(\frac{n!\,n!}{(n+1)!\,(n+1)!}\right)$$

$$= \lim_{n\to\infty} \left(\frac{\cancel{n!}\,\cancel{n!}}{(n+1)(n+1)\cancel{n!}\,\cancel{n!}}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{(n+1)^2}\right) = 0.$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n!)^2}$ converges by Ratio Test.

5. [2pts.] Give the definition for convergence of $s = a_1 + a_2 + a_3 + \cdots$ in terms of an explicit limiting process.

$$s = \sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \left(\sum_{k=1}^{n} a_k \right).$$

6.[3pts] Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Explain in a sentence or two what it means for the limit of the sequence to be equal to 3. Also, give an example of a sequence which converges to 3.

$\lim_{n \to \infty} a_n = 3$ means that as n attains arbitrarily large values the values for a_n become arbitrainly close to 3.

$a_n = 3$ has $\lim_{n \to \infty} a_n = 3$.

7. [3pts] List the first three terms in the sequence of partial sums for $\sum_{n=1}^{\infty} \frac{1}{n!}$.

$$S_1 = \frac{1}{1!}$$

$$S_2 = a_1 + a_2 = 1 + \frac{1}{2!}$$

$$S_3 = a_1 + a_2 + a_3 = 1 + \frac{1}{2!} + \frac{1}{3!}$$

$$\{S_n\}_{n=1}^{\infty} = \left\{1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{6}, \ldots\right\}$$