Math 132, Test 3, March 31, 2009 Name:

No graphing calculators or electronic communication of any kind. If you need extra paper please ask. Credit will be awarded for correct content and clarity of presentation. Prepare for math battle. This test has 100 points. There is a take-home bonus problem at the end with instructions. Make sure to at least attempt each part except the bonus.

1. [10pts.] Find the Interval of Convergence(IOC) and Radius of Convergence (ROC) for the power series $f(x)$ given below: (justify your answer)

$$f(x) = \sum_{n=0}^{\infty} 3^n (x-2)^n$$ this is geometric with $r = 3(x-2)$.

Observe $|r| < 1 \Rightarrow |3(x-2)| < 1$

$\Rightarrow |x-2| < 1/3$

$\Rightarrow -1/3 < x-2 < 1/3 \Rightarrow \boxed{R.O.C. = 1/3}$

$\Rightarrow 2 - 1/3 < x < 2 + 1/3$

$\Rightarrow 5/3 < x < 7/3$

$\Rightarrow \boxed{I.O.C. = (5/3, 7/3)}$

2.[5pts] Consider the function $f(x) = \frac{1}{x-2}$. What is the largest IOC possible for a power series representation of $f(x)$ centered at $a = 2.5$.

Notice $2 \notin dom(f)$. At best we can approach $x=2$ for the I.O.C. which must be centered at 2.5 thus

$\boxed{(2,3) = \text{maximal I.O.C.}}$

3.[10pts.] Find the IOC and ROC for the power series given below: (again justify your answer with appropriate tests as was done in lecture)

$g(x) = \sum_{n=1}^{\infty} \frac{1}{n}(x-2)^n$ identify $a_n = \frac{(x-2)^n}{n}$ and use ratio test,

$$L = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{n+1} \frac{n}{(x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left| (x-2) \frac{n}{n+1} \right|$$

$$= |x-2| \cancel{\lim_{n\to\infty} \left(\frac{n}{n+1}\right)} \to 1$$

$$= |x-2|$$

Thus $L < 1$ iff $|x-2| < 1 \Rightarrow -1 < x-2 < 1$

$\Rightarrow 1 < x < 3$

$\Rightarrow (1,3) \subset$ I.O.C.

Consider endpts. $x=1$ & $x=3$:

$g(1) = \sum_{n=1}^{\infty} \frac{1}{n}(1-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$: converges by A.S.T. this is alternating harmonic series

$g(3) = \sum_{n=1}^{\infty} \frac{1}{n}(3-2)^n = \sum_{n=1}^{\infty} \frac{1}{n}$: diverges by $p=1$ series test.

To conclude $\boxed{\text{I.O.C.} = [1,3), \quad \text{R.O.C.} = 1}$

4.[4pts] Given the power series $\sum_{n=0}^{\infty} c_n(x-1)^n$ diverges at $x=3$.

(a.) is it possible for $\sum_{n=0}^{\infty} c_n(x-1)^n$ to converge at $x=2$?

The biggest I.O.C. is $(1-R, 1+R)$ such that $1+R=3$ thus $R=2$ and $(-1,3)$

So YES it could converge

(b.) is it possible for $\sum_{n=0}^{\infty} c_n(x-1)^n$ to converge at $x=-3$?

No, this is outside the largest I.O.C.

(If $x=-3$ was included then I.O.C. $\neq (a+R, a-R)$ and our $Th^{\underline{m}}$ would be contradicted)

5.[2pts] What is $T_1(x)$ centered at $x=a$ for the function $f(x)$? Describe the geometric significance of the n=1 Taylor polynomial.

$T_1(x) = f(a) + f'(a)(x-a)$ this is the linearization of $y=f(x)$ at $x=a$. Its graph is the tangent line at $x=a$.

6. [9pts] Find the complete power series representation centered about zero for the function given below, state the IOC for the series.

$$f(x) = \frac{3}{1+x^2} = \sum_{n=0}^{\infty} 3(-x^2)^n$$

$$= \boxed{\sum_{n=0}^{\infty} 3(-1)^n x^{2n} \quad \text{for I.O.C.} = (-1,1)}$$

7.[10pts] Find the complete power series centered at $a = 0$ for the function $f(x) = \tan^{-1}(x^3) + 1$.

$$\frac{df}{dx} = \frac{3x^2}{1+x^6} = \sum_{n=0}^{\infty} 3x^2(-x^6)^n$$

geometric series with $a = 3x^2$, $r = -x^6$

$$\frac{df}{dx} = \sum_{n=0}^{\infty} 3(-1)^n x^{6n+2}$$

$$f(x) = \int \frac{df}{dx}\,dx = C + \sum_{n=0}^{\infty} \frac{3(-1)^n}{6n+3} x^{6n+3}$$

Notice $f(0) = \tan^{-1}(0) + 1 = 1$ and $\underbrace{f(0) = C}_{}$ $= 1$

Hence,

notice $n=0$ has x^3 and $n \geq 1$ has x^{6n+3}, these all vanish when evaluated at $x = 0$.

$$\boxed{f(x) = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{6n+3}}$$

8.[10pts] Find the complete power series solution for the following indefinite integral

$$\int x^2 \cos(x^2)\,dx = \int \left(x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x^2)^{2n} \right) dx$$

$$= \int \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n+2} \right) dx$$

$$= \boxed{C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\,(4n+3)} x^{4n+3}}$$

9.[25pts] Find the first three non-zero terms in the power series centered at a for the functions given below

(a.) $f(x) = \dfrac{x^5}{\sqrt{2-4x}}$ find power series representation centered at $a = 0$.

$$= \frac{x^5}{\sqrt{2(1-2x)}}$$

$$= \frac{x^5}{\sqrt{2}}(1-2x)^{-1/2}$$

$$= \frac{x^5}{\sqrt{2}}\left(1 + ku + \tfrac{1}{2}k(k-1)u^2 + \cdots\right) \qquad k = -\tfrac{1}{2},\ u = -2x$$

$$= \frac{x^5}{\sqrt{2}}\left(1 - \tfrac{1}{2}(-2x) + \tfrac{1}{2}\left(\tfrac{-1}{2}\right)\left(\tfrac{-3}{2}\right)(-2x)^2 + \cdots\right)$$

$$= \boxed{\frac{1}{\sqrt{2}}\left(x^5 + x^6 + \tfrac{3}{4}x^7 + \cdots\right)}$$

(b.) find the power series centered at $a = 0$ for $f(x) = x\sin(2x)$

$$f(x) = x\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2x)^{2n+1}$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\,2^{2n+1}\,x^{2n+2}}$$

$$= x\left(2x - \tfrac{1}{3!}(2x)^3 + \tfrac{1}{5!}(2x)^5 + \cdots\right)$$

$$= 2x^2 - \frac{8}{3!}x^4 + \frac{32}{5!}x^6 + \cdots \qquad \frac{32}{5\cdot4\cdot3\cdot2} = \frac{4}{15}$$

$$= \underbrace{\boxed{2x^2 - \tfrac{4}{3}x^4 + \tfrac{4}{15}x^6 + \cdots}}_{\text{this was all I asked for.}}$$

(c.) find the power series centered at $a = 0$ for $f(x) = \sin(x+3)$

$$\sin(x+3) = \sin(x)\cos(3) + \cos(x)\sin(3)$$

$$= \left(x - \frac{1}{3!}x^3 + \cdots\right)\cos(3) + \left(1 - \frac{x^2}{2!} + \cdots\right)\sin(3)$$

$$= \boxed{\sin(3) + \cos(3)\cdot x - \sin(3)\frac{x^2}{2} + \cdots}$$

(d.) find the power series centered at $a = 0$ for $f(x) = (3 + x^2)e^{x+2}$

$$f(x) = (3+x^2)e^{x+2}$$

$$= (3+x^2)e^x e^2$$

$$= (3+x^2)\left(1 + x + \frac{x^2}{2} + \cdots\right)e^2$$

$$= e^2\left(3 + 3x + \frac{3}{2}x^2 + x^2 + \cdots\right)$$

$$= \boxed{e^2\left(3 + 3x + \frac{5}{2}x^2 + \cdots\right)}$$

(e.) find the power series centered at $a = 1$ for $f(x) = x^2$

$f(x) = x^2$ $\quad f(1) = 1$

$f'(x) = 2x$ $\quad f'(1) = 2$

$f''(x) = 2$ $\quad f''(1) = 2$

$f'''(x) = 0$

$f^{(n)}(x) = 0$ for $n \geq 3$

$$f(x) = x^2 = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \cdots$$

$$= \boxed{1 + 2(x-1) + (x-1)^2 = f(x)}$$

10.[5pts] Find a bound $M > 0$ such that $|error| \leq M$ given that the error is defined implicitly in the equation below:

$$\int_0^1 x^4 \cos(x)\, dx = \frac{1}{5} + error$$

$$\int_0^1 x^4 \cos(x)\,dx = \int_0^1 x^4\left(1 - \frac{1}{2}x^2 + \cdots\right)dx$$

$$= \int_0^1 \left(x^4 - \frac{1}{2}x^6 + \cdots\right)dx$$

$$= \left(\frac{1}{5}x^5 - \frac{1}{14}x^7 + \cdots\right)\Big|_0^1$$

$$= \underbrace{\frac{1}{5} - \frac{1}{14} + \cdots}$$

Alternating series $\Rightarrow$ $\boxed{|error| \leq \frac{1}{14}}$

11.[7pts] Use Taylor's Theorem to find the first 4 terms in the Taylor series for $f(x) = \frac{\ln(x)}{x}$ centered around $a = 1$. In other words, find the $T_3(x)$ centered about $a = 1$ for the function $f(x) = \frac{\ln(x)}{x}$.

$$f(x) = \frac{\ln(x)}{x} \quad : \quad f(1) = 0$$

$$f'(x) = \frac{-1}{x^2}\ln(x) + \frac{1}{x^2} \quad : \quad f'(1) = 1$$

$$f''(x) = \frac{2}{x^3}\ln(x) - \frac{1}{x^3} - \frac{2}{x^3} \quad : \quad f''(1) = -3$$

$$f'''(x) = \frac{-6}{x^4}\ln(x) + \frac{2}{x^4} + \frac{3}{x^4} + \frac{6}{x^4} \quad : \quad f'''(1) = 11$$

$$\boxed{T_3(x) = (x-1) - \frac{3}{2}(x-1)^2 + \frac{11}{3!}(x-1)^3}$$

12. [2pts.] Give the definition for convergence of $s = a_1 + a_2 + a_3 + \cdots$ in terms of an explicit limiting process.

$$s = \lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k\right).$$

13. [1pts] List the first three terms in the sequence of partial sums for $\sum_{n=1}^{\infty} n^2$.

$$\{S_n\}_{n=1}^{\infty} = \{1, 1+4, 1+4+9, \ldots\}$$
$$= \{1, 5, 14, \ldots\}.$$

TAKE-HOME BONUS PROBLEM(s):

(return to my office box at or before 8:50am April 6.)

1. [2pts] Calculate the Taylor series for $e^{i\theta}$ where θ is a real variable and $i^2 = -1$.

2. [2pts] Ignore the fact we already have a proof of $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Rederive Euler's identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ from the series found in 1.

3. [2pts] Multiply the series given below:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

Simplify the answer, and identify the resulting series. Does this make sense?

4.[2pts] If S_n is the n^{th} partial sum for $\sum_{n=0}^{\infty} a_n$ and $\overline{S}_n$ is the n^{th} partial sum for $\sum_{n=0}^{\infty} ca_n$, then note that $\overline{S}_n = cS_n$ for all $n \in \mathbb{N} \cup \{0\}$. Let $c \in \mathbb{R}$, prove that if $\sum_{n=0}^{\infty} a_n$ converges then $\sum_{n=0}^{\infty} ca_n$ converges. (your proof should flow from the definition of convergence for series)