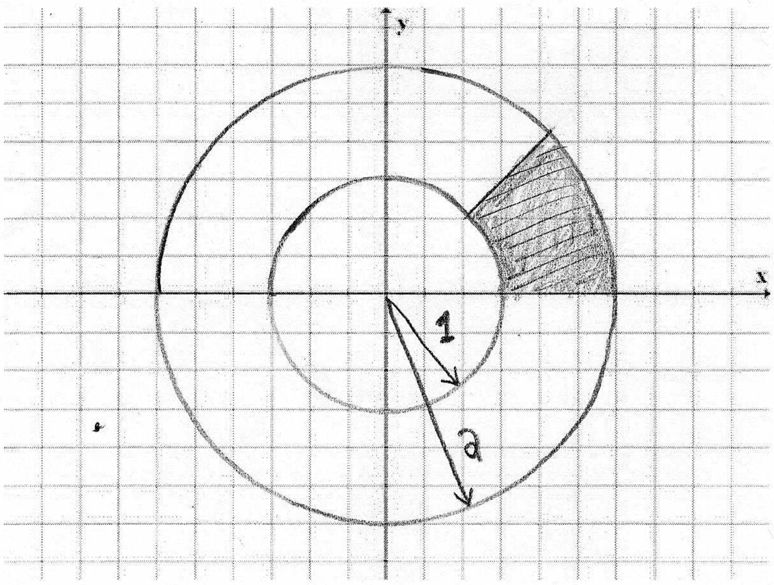
Math 132, Test 4, April 28, 2009 Name: .

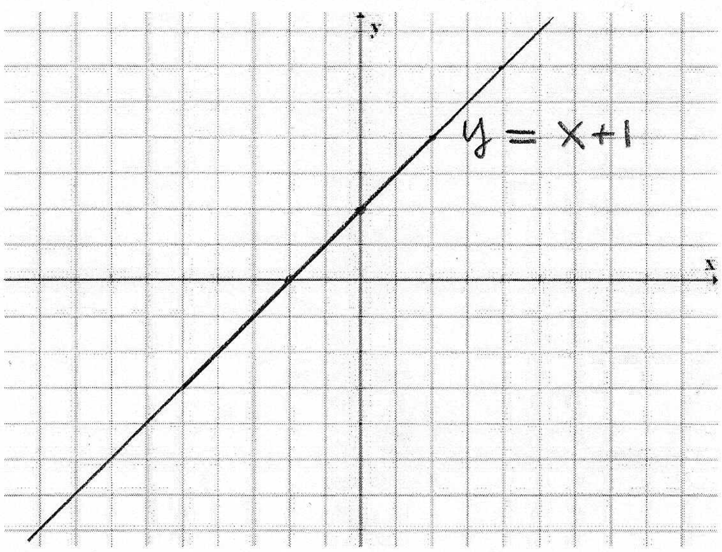
No graphing calculators or electronic communication of any kind. If you need extra paper please ask. Credit will be awarded for correct content and clarity of presentation. Prepare for math battle. This test has 100 points and 5 pts bonus (see the white-board for the bonus).

1. [20pts.] Let r, θ be polar coordinates. Complete the following tasks:

a.) sketch the region in the plane which has $1 \leq r \leq 2$, and $0 \leq \theta \leq \pi/4$



b.) Find the Cartesian coordinate form of the equation $r\sin(\theta) = r\cos(\theta) + 1$,



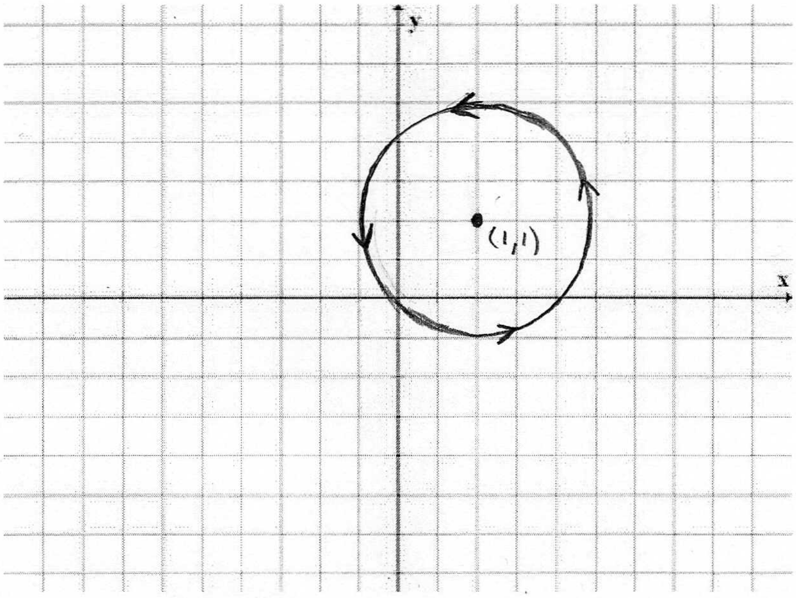
$x = r\cos\theta$

$y = r\sin\theta$

therefore, $r\sin\theta = r\cos\theta + 1$

$\Rightarrow \boxed{y = x + 1}$

c.) Find the Cartesian coordinate form of the equation $r = 2\sin(\theta) + 2\cos(\theta)$,



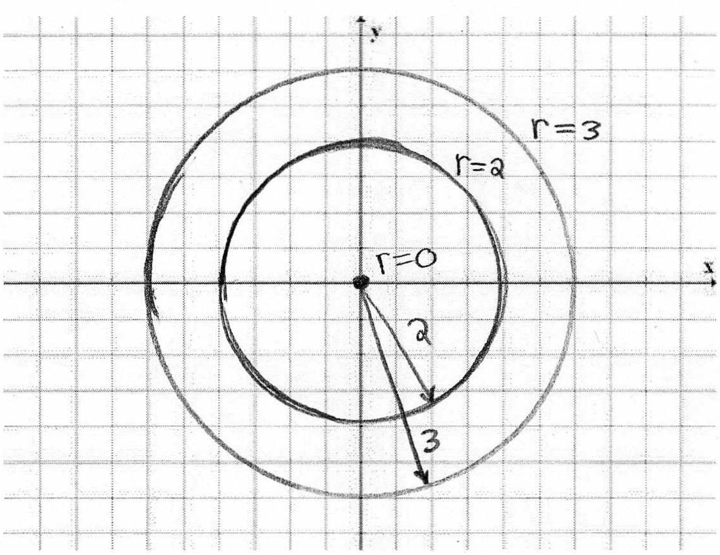
$\Rightarrow r^2 = 2r\sin\theta + 2r\cos\theta$

$x^2+y^2 = 2y + 2x$

$$\boxed{(x-1)^2+(y-1)^2 = 2}$$

circle, radius $\sqrt{2}$ centered at $(1,1)$.

d.) Plot the graph of $r^3 + 6r = 5r^2$



$r^3 + 6r = 5r^2$

$\Rightarrow r^3 + 6r - 5r^2 = 0$

$\Rightarrow r(r^2 - 5r + 6) = 0$

$r(r-3)(r-2) = 0$

$\underline{r=0,\ r=3,\ r=2}$

2.[12pts] Eliminate the parameter t to find the corresponding Cartesian equation for the curves with the following parametric descriptions,

a.) $x = 1 + 3t,\ y = 2 - t^2$ where $t \in \mathbb{R}$.

$$t = \frac{x-1}{3} \Rightarrow y = 2 - \left(\frac{x-1}{3}\right)^2$$

$$\Rightarrow \boxed{y = -\frac{1}{9}(x-1)^2 + 2}$$

(parabola with vertex $(1,2)$. opens down)

or

$$\boxed{y = \frac{-1}{9}x^2 + \frac{2}{9}x + \frac{17}{9}}$$

b.) $x = 1 + 4\cos(t),\ y = 5\sin(t)$ where $-\pi/2 \leq t \leq \pi/2$.

$$\frac{x-1}{4} = \cos t \quad \& \quad \frac{y}{5} = \sin t$$

and $\cos^2 t + \sin^2 t = 1$

$$\Rightarrow \boxed{\frac{(x-1)^2}{16} + \frac{y^2}{25} = 1}$$

for $-\pi/2 \leq t \leq \pi/2$ we know $-1 \leq \sin t \leq 1$ while $0 \leq \cos t \leq 1$. (look at the graphs of sine and cosine)

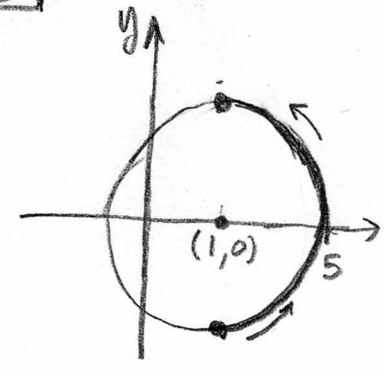
It follows $-5 \leq \underbrace{5\sin t}_{y} \leq 5$ and $1 \leq \underbrace{1 + 4\cos t}_{x} \leq 5$

therefore, $\boxed{1 \leq x \leq 5 \text{ and } -5 \leq y \leq 5}$

c.) $x = \sinh(t),\ y = \cosh(t)$ where $t \in \mathbb{R}$.

$$\cosh^2 t - \sinh^2 t = 1$$

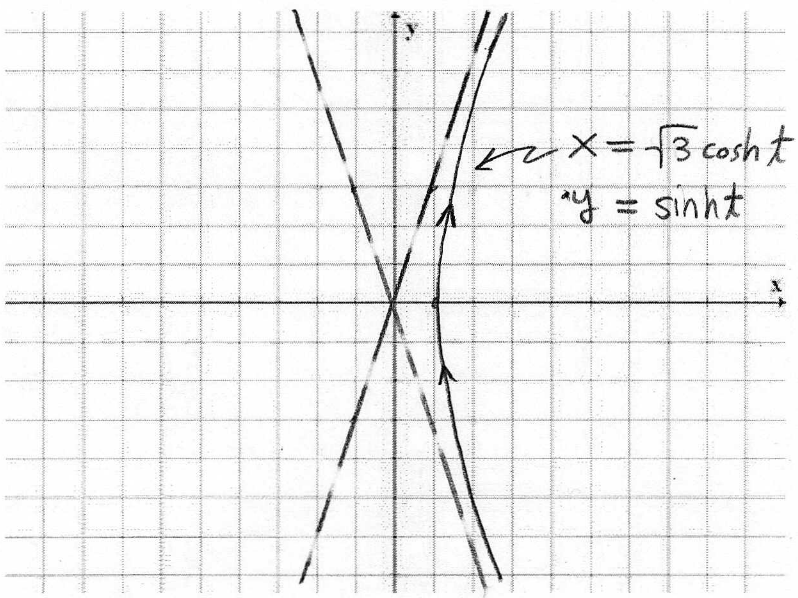
$$\therefore \boxed{y^2 - x^2 = 1}$$



(inequalities do check with the geometry)

3.[20pts.] Find parametric equations to describe the curves given below. Assume the graphs are oriented counter-clockwise (except for part c.). Also provide a sketch of the curve.

a.) $9x^2 - 3y^2 = 3$, for $x \geq 1$



$$\underbrace{3x^2}_{\cosh^2(t)} - \underbrace{y^2}_{\sinh^2(t)} = 1$$

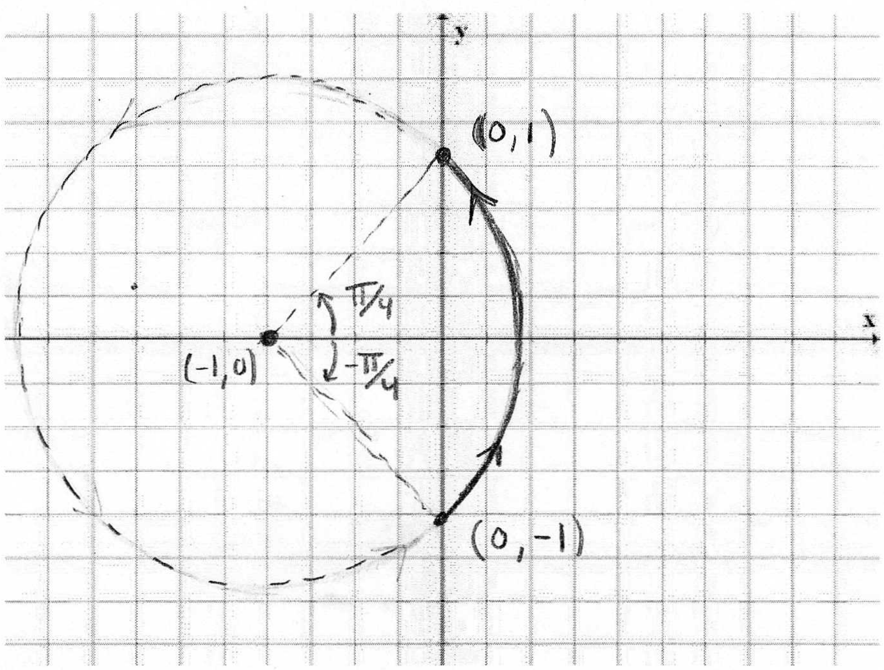
choose,

$$\boxed{\begin{array}{l} x = \frac{1}{\sqrt{3}}\cosh t \\ y = \sinh t \\ \text{for } t \in \mathbb{R} \end{array}}$$

asymptotes,

$$9x^2 = 3y^2$$
$$y = \pm 3x$$

b.) $x^2 + 2x + y^2 = 1$, for $x \geq 0$



$$\underbrace{(x+1)^2 + y^2 = 2}$$

radius $\sqrt{2}$ centered at $(-1,0)$.

$$\boxed{\begin{array}{l} x = 1 + \sqrt{2}\cos\beta \\ y = \sqrt{2}\sin\beta \\ -\frac{\pi}{4} \leq \beta \leq \frac{\pi}{4} \end{array}}$$

5.[13pts] Calculate the arclength of **ONE** of the following curves (your choice)

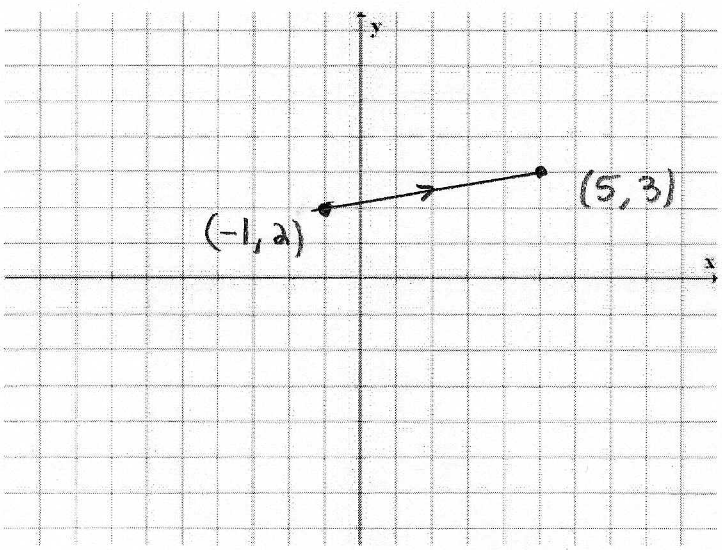
(a.) $y = \ln(\cos(x))$, for $0 \le x \le \pi/3$ ← Solved in hwk sol^{n}

(b.) $x = \cos^2(t)$, $y = \cos(t)$, for $0 \le t \le 4\pi$

(c.) $x = 3t - t^3$, $y = 3t^2$, the loop at the origin

(d.) $x = 3\cos(t) - \cos(3t)$, $y = 3\sin(t) - \sin(3t)$, for $0 \le t \le \pi$

c.) line segment from (-1, 2) to (5, 3).



$$\vec{r}(t) = (1-t)(-1,2) + t(5,3)$$
$$= (-1+t+5t,\ 2-2t+3t)$$
$$= (\underbrace{-1+6t}_{x},\ \underbrace{2+t}_{y})$$

$$\boxed{\begin{aligned} x &= -1+6t \\ y &= 2+t \\ &\text{for } 0 \le t \le 1 \end{aligned}}$$

4.[5pts] Find a value for the constant c which makes $f(x) = \dfrac{c}{1+x^2}$ a probability density function.

We need $f(x) \geq 0 \ \forall x \in \mathbb{R}$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} \frac{c\,dx}{1+x^2} = \int_{-\infty}^{0} \frac{c\,dx}{1+x^2} + \int_{0}^{\infty} \frac{c\,dx}{1+x^2}$$

$$= \lim_{t \to -\infty} \left(c\tan^{-1}(x) \Big|_t^0 \right) + \lim_{s \to \infty} \left(c\tan^{-1}(x) \Big|_0^s \right)$$

$$= c \lim_{t \to -\infty} \left(-\tan^{-1}(t) \right) + c \lim_{s \to \infty} \left(\tan^{-1}(s) \right)$$

$$= \frac{c\pi}{2} + \frac{c\pi}{2}$$

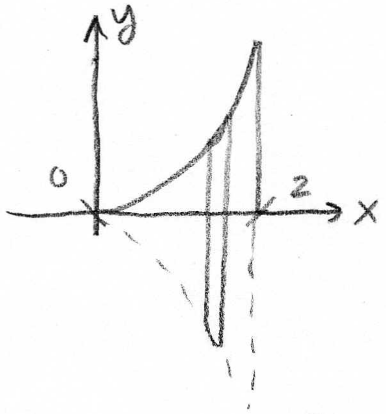
$$= c\pi = 1 \Rightarrow \boxed{c = \frac{1}{\pi}}$$

Remark: could argue by symmetry

$$\int_{-\infty}^{\infty} \frac{c}{1+x^2}\,dx = 2\int_0^{\infty} \frac{c\,dx}{1+x^2}$$

would save some writing

6.[10pts] Find the surface area of the surface of revolution formed by rotating $y = x^3,\ 0 \le x \le 2$ around the x-axis.



$$dA = 2\pi r\, ds$$
$$= 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$
$$= 2\pi x^3 \sqrt{1+9x^4}\, dx$$

$$A = \int_0^2 2\pi x^3 \sqrt{1+9x^4}\, dx$$

$u = 1+9x^4$

$du = 36x^3\, dx$

$2x^3 dx = \frac{du}{18}$

$$= \frac{\pi}{18}\frac{2}{3}\left(1+9x^4\right)^{3/2}\Big|_0^2$$
$$= \frac{\pi}{27}\left(\left(1+144\right)^{3/2} - 1\right)$$
$$= \boxed{\frac{\pi}{27}\left(145^{3/2} - 1\right)}$$

7.[5pts] Find the average of $f(x) = \cos^2(x)$ over the interval $[0, \pi]$.

$$f_{avg} = \frac{1}{\pi - 0}\int_0^{\pi} \cos^2(x)\, dx$$
$$= \frac{1}{\pi}\int_0^{\pi} \frac{1}{2}\left(1 - \cos(2x)\right) dx$$
$$= \frac{1}{\pi}\left(\frac{x}{2} - \frac{1}{4}\underbrace{\sin(2x)}_{0}\right)\Big|_0^{\pi}$$
$$= \boxed{\frac{1}{2}}$$

8. [15pts] Solve $y'' + 5y' + 6y = t^2 + 2$.

$$\lambda^2 + 5\lambda + 6 = 0$$
$$(\lambda+3)(\lambda+2) = 0$$
$$\lambda_1 = -3, \quad \lambda_2 = -2 \qquad \therefore \quad \underline{y_h = C_1 e^{-3t} + C_2 e^{-2t}}$$

Guess $y_p = At^2 + Bt + C$

$$y_p' = 2At + B$$
$$y_p'' = 2A$$

Substitute,

$$y_p'' + 5y_p' + 6y_p = t^2 + 2$$
$$2A + 5(2At + B) + 6(At^2 + Bt + C) = t^2 + 2$$
$$2A + 5B + 6C + t(10A + 6B) + 6At^2 = t^2 + 2$$

t^2 : $1 = 6A \longrightarrow A = 1/6$

t : $0 = 10A + 6B \longrightarrow B = -\frac{1}{6}(10A) = -\frac{10}{36} = B$

t^0 : $2 = \underbrace{2A + 5B + 6C}$

$$C = \frac{1}{6}(2 - 2A - 5B)$$
$$= \frac{1}{6}\left(2 - \frac{2}{6} + \frac{25}{18}\right)$$
$$= \frac{1}{6}\left(\frac{36 - 6 + 25}{18}\right) = \frac{55}{6(18)} = \frac{55}{108}$$

$$\boxed{y = C_1 e^{-3t} + C_2 e^{-2t} + \frac{1}{6}t^2 - \frac{5}{18}t + \frac{55}{108}}$$