Concerning the expansion of $\frac{1}{(1+x)^2} = f(x)$

We have a good way to find the power series centered about zero. Integrate, apply geometric series result, differentiate and find power series rep. for $f(x)$ with an I.O.C. of $(-1,1)$.

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Another way to try is to multiply the denominator $(1+x)^2 = 1 + 2x + x^2 = 1-(-x(x+2))$. Thus

$$\frac{1}{(1+x)^2} = \underbrace{\frac{1}{1-(-x(x+2))} = \sum_{n=0}^{\infty}(-1)^n[x(x+2)]^n}_{\text{provided } |r| = |x(x+2)| < 1}$$

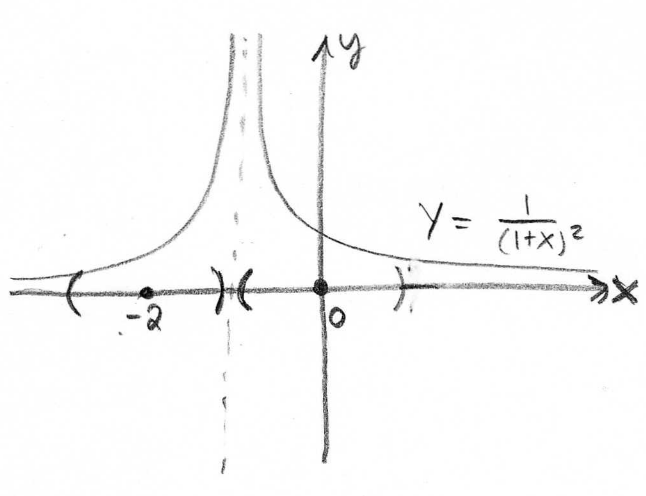
Notice the RHS is <u>not</u> a power series. (need $\sum_{n=0}^{\infty} c_n(x-a)^n$ and we have sort-of two centers, namely $x=0$ or $x=-2$)

What does $|r| < 1$ entail here?

$$\begin{aligned}
|x(x+2)| < 1 &\iff -1 < x(x+2) < 1 \\
&\iff 0 < x(x+2)+1 < 2 \\
&\iff 0 < x^2+2x+1 < 2 \\
&\iff 0 < (x+1)^2 < 2 \\
&\iff 0 < x+1 < 2 \quad \text{or} \quad -2 < x+1 < 0 \\
&\iff \underbrace{-1 < x < 1 \quad \text{or} \quad -3 < x < -1}_{\text{these are disconnected.}}
\end{aligned}$$

$(-1,1)$ and $(-3,-1)$

they both edge-up to the V.A. $x=-1$ for $f(x) = \frac{1}{(1+x)^2}$.



Why are there two possible regions where $|r| < 1$. Either $(-1,1)$ or $(-3,-1)$ would seem to be possible I.O.C.'s for our geometric series. How is this possible? Consider,

$$\frac{1}{(1+x)^2} = 1 - x(x+2) + x^2(x+2)^2 - x^3(x+2)^3 + \cdots$$

$$= 1 - x^2 - 2x + x^2(x^2+4x+4) - x^3(x^3+6x^2+12x+8) + \cdots$$

$$= \underbrace{1 - x^2 - 2x + x^4 + 4x^3 + 4x^2 - x^6 - 6x^5 - 12x^4 - 8x^3 + \cdots}_{\substack{\text{all terms up to order } x^3 \\ \text{only \underline{some} of the order } x^4, x^5, x^6}}$$

$$= \underbrace{1 - 2x + 3x^2 - 4x^3 + \cdots}_{\substack{\text{this is a power series centered} \\ \text{at zero. We can show its I.O.C} = (-1,1).}}$$

<u>In contrast</u>, we could rearrange terms as to preserve the $(x+2)$ structure,

$$\frac{1}{(1+x)^2} = 1 - x(x+2) + x^2(x+2)^2 + \cdots$$

$$= 1 - x^2 - 2x + x^2(x^2+4x+4) + \cdots$$

$$= 1 - 2(x+2) + 4 - \underline{x}^2 + x^4 + 4x^3 + \underline{4x}^2 + \cdots$$

$$= 1 - 2(x+2) + 3(\underline{x}^2 + 4x + 4) - 12x - 8 + \cdots$$

$$= \underbrace{1 - 2(x+2) + 3(x+2)^2}_{\text{everything upto 2nd order}} + \underbrace{4x^3 - 12x - 8 - x^3(x+2)^3}_{\substack{\text{goes into making} \\ -4(x+2)^3}} + \cdots$$

The next term will be $-4(x+2)^3$. Observe,

$$4x^3 - 12x - 8 - x^3(x+2)^3 = 4x^3 - 12x - 8 - x^3(x^3+6x^2+12x+8)$$

$$= -4x^3 - 12x - 8 + \mathcal{O}(x^4)$$

$$= -4(x^3+6x^2+12x+8) + 24 + 36x + 24x^2 + \cdots$$

$$= -4(x+2)^3 + \cdots$$

<u>Conclusion</u>: Can rearrange geom. series <u>two</u> ways to obtain center at 0 or -2