LECTURE 10: LINEAR TRANSFORMATIONS ON COLUMN VECTORS

A function T for which $T(v+w) = T(v) + T(w)$ is called <u>additive</u> and if $T(cv) = cT(v)$ for all scalars c then T is called homogeneous or "it allows scalars to pull out". Such a function is said to be <u>linear</u>.

Def^n/ If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a function such that

(1.) $T(x+y) = T(x) + T(y)$ for all $x, y \in \mathbb{R}^n$

(2.) $T(cx) = cT(x)$ for all $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$

then T is a <u>linear</u> <u>transformation</u> and we write $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \leftarrow$ set of all such linear trans.

$\boxed{E1}$ $T(x_1, x_2) = x_1 x_2$ is not linear since

$$T(cx) = T(cx_1, cx_2) = (cx_1)(cx_2) = c^2 x_1 x_2 = c^2 T(x)$$

$$\begin{aligned} T(x+y) &= T((x_1+y_1, x_2+y_2)) \\ &= (x_1+y_1)(x_2+y_2) \\ &= \underbrace{x_1 x_2}_{T(x)} + \underbrace{x_1 y_2 + y_1 x_2}_{\substack{\text{shouldn't} \\ \text{be here} \\ \text{for } T \text{ to be linear.}}} + \underbrace{y_1 y_2}_{T(y)} \end{aligned}$$

$\boxed{E2}$ Let $A \in \mathbb{R}^{m \times n}$ and let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be defined by $T(x) = Ax$ for all $x \in \mathbb{R}^n$

$$T(x+y) = A(x+y) = Ax + Ay = T(x) + T(y)$$

$$T(cx) = A(cx) = cAx = cT(x)$$

thus T is a <u>linear</u> <u>transformation</u> from $\mathbb{R}^n$ to $\mathbb{R}^m$.

(this example shows $L_A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, it's linear)

Def^n/ $L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is <u>LEFT</u> <u>MULTIPLICATION</u> by $A \in \mathbb{R}^{m \times n}$ defined by $L_A(x) = Ax$ for $x \in \mathbb{R}^n$

Proposition: If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a function whose formula can be expressed as a matrix multiplication $T(v) = Av$ for all $v \in \mathbb{R}^n$ then T is a <u>linear</u> <u>transformation</u>

$\boxed{E3}$ $T(x,y,z) = (x+2y,\ 3y+z)$, $T: \mathbb{R}^3 \to \mathbb{R}^2$

$$= \begin{bmatrix} x+2y \\ 3y+z \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}}_{[T]} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \therefore\ T \text{ linear mapping}$$

$[T] \leftarrow$ standard matrix of T is 2×3 here.

Remark: $T = L_{[T]}$, T is LEFT MULTIPLICATION BY $[T]$.

$\boxed{E4}$ $T(x,y) = (3x-8y,\ 2x+y)$, $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$= \begin{bmatrix} 3x-8y \\ 2x+y \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 3 & -8 \\ 2 & 1 \end{bmatrix}}_{[T] \text{ is } 2\times 2 \text{ matrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\boxed{E5}$ $T: \mathbb{R}^4 \to \mathbb{R}^3$ defined by,

$$T(x) = \begin{bmatrix} x_1 - 2x_4 \\ x_1 + x_2 + x_3 + x_4 \\ 3x_1 - 2x_2 - 7x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & -2 \\ 1 & 1 & 1 & 1 \\ 3 & -2 & -7 & 0 \end{bmatrix}}_{[T] \in \mathbb{R}^{3\times 4}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Thm/ A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if $T(x) = [T]x$ for all x in $\mathbb{R}^n$ where $[T] = [T(e_1) | T(e_2) | \cdots | T(e_n)] \in \mathbb{R}^{m \times n}$

Proof: we showed $\Leftarrow$ direction in $\boxed{E2}$. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and let $x = \sum_{i=1}^{n} x_i e_i$ then by linearity of T it follows that

$$
\begin{aligned}
T(x) &= T\left(\sum_{i=1}^{n} x_i e_i\right) \\
&= \sum_{i=1}^{n} x_i T(e_i) \\
&= x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n) \\
&= [T(e_1) | T(e_2) | \cdots | T(e_n)] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= [T]x. //
\end{aligned}
$$

Defn/ If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping then the <u>standard matrix</u> of T is $[T]$ and it is defined by $[T] = [T(e_1) | T(e_2) | \cdots | T(e_n)]$.

$\boxed{E6}$ $T(x) = x$ for each $x \in \mathbb{R}^n$ defines the <u>identity</u> transformation on $\mathbb{R}^n$ denoted $T = Id_{\mathbb{R}^n}$ or simply $T = Id$ when context is clear. Observe

$$
\begin{aligned}
[Id] &= [Id(e_1) | Id(e_2) | \cdots | Id(e_n)] \\
&= [e_1 | e_2 | \cdots | e_n] \\
&= I_n
\end{aligned}
$$

We build new transformations from old by the usual pointwise rules, just like in precalculus,

Def$^{\underline{n}}$/ Given $S, T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$ we define $(S \pm T)(x) = S(x) \pm T(x)$ and for $c \in \mathbb{R}$, $(cT)(x) = c\,T(x)$ and the <u>composite</u> $(T \circ U)(x) = T(U(x))$ for each x in $\mathbb{R}^p$

We can argue that linearity of S, T and U imply that $S+T$, $S-T$, cT and $T \circ U : \mathbb{R}^p \to \mathbb{R}^n \to \mathbb{R}^m$ are all likewise linear transformations.

Th$^{\underline{m}}$/ For S, T, U as above and $c \in \mathbb{R}$,

① $[S+T] = [S] + [T]$

② $[S-T] = [S] - [T]$

③ $[cT] = c[T]$

④ $[T \circ U] = [T][U]$

<u>Proof:</u> I leave ①, ② and ③ to the reader. Let's work through ④ where $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$ hence $[T] \in \mathbb{R}^{m \times n}$, $[U] \in \mathbb{R}^{n \times p}$

thus $[T][U]$ is $(m \times n)(n \times p)$ which yields $m \times p$ as it ought since $T \circ U : \mathbb{R}^p \to \mathbb{R}^m$. Consider,

$$\begin{aligned}
[T \circ U] &= [T(U(e_1)) | \cdots | T(U(e_p))] && : \text{def}^{\underline{n}} \text{ of } [\ ] \\
&= [[T]U(e_1) | \cdots | [T]U(e_p)] && : T(v) = [T]v \\
&= [T][U(e_1) | \cdots | U(e_p)] && : \text{column-by-col. mult. rule} \\
&= [T][U]. // && : \text{def}^{\underline{n}} \text{ of } [U].
\end{aligned}$$

$\boxed{E7}$ $T(x,y) = (x+y,\ x-y)$, $T : \mathbb{R}^2 \to \mathbb{R}^2$

$U(x,y,z) = (3x-4y+z,\ 2y-13z)$, $U : \mathbb{R}^3 \to \mathbb{R}^2$

$$(T \circ U)(x,y,z) = T(3x-4y+z,\ 2y-13z)$$
$$= \big((3x-4y+z)+(2y-13z),\ (3x-4y+z)-(2y-13z)\big)$$
$$= (3x-2y-12z,\ 3x-6y+14z)$$
$$= \underbrace{\begin{bmatrix} 3 & -2 & -12 \\ 3 & -6 & 14 \end{bmatrix}}_{[T \circ U]} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

However, $[T] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $[U] = \begin{bmatrix} 3 & -4 & 1 \\ 0 & 2 & -13 \end{bmatrix}$

thus $[T][U] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -4 & 1 \\ 0 & 2 & -13 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -12 \\ 3 & -6 & 14 \end{bmatrix}$

Therefore, $[T \circ U] = [T][U]$ as claimed.

Thm/ Given $T : \mathbb{R}^n \to \mathbb{R}^n$ a linear transformation,

(1.) $T^n = T \circ T^{n-1}$, $T^0 = Id$ for $n \in \mathbb{N}$ then $[T^n] = [T]^n$

(2.) If T is invertible with inverse T^{-1} then $[T^n] = [T]^n$ for $n \in \mathbb{Z}$ where for instance $T^{-2} = (T^{-1})^2 = T^{-1} \circ T^{-1}$ etc.

Proof: If $T \circ T^{-1} = Id$ then $[T \circ T^{-1}] = [T][T^{-1}] = [Id]$

thus $[T][T^{-1}] = I \Rightarrow [T^{-1}] = [T]^{-1}$, the rest I leave to you. //