Lecture 11: Linear Transformations & Vector Spaces

If T is a linear transformation then $T(0+0) = T(0)+T(0) = T(0)$ thus $T(0) = 0$. Suppose $T(x) = T(y)$ then $T(x) - T(y) = 0$ and so $T(x-y) = 0$. Thus if we wish $T(x) = T(y) \Rightarrow x = y$ then we need that $T(z) = 0 \Leftrightarrow z = 0$.

Def^{n}/ $\text{Ker}(T) = \{x \mid T(x) = 0\}$

$\text{Range}(T) = \{T(x) \mid x \in \text{dom}(T)\}$

For $T : \mathbb{R}^n \to \mathbb{R}^m$ we can calculate the kernel and image or range of T via standard matrixes,

$$\text{Ker}(T) = \text{Null}([T]) = \{x \in \mathbb{R}^n \mid [T]x = 0\}$$

$$\text{Range}(T) = \text{Col}([T]) = \text{span}\{\text{col}_1[T], \dots, \text{col}_n[T]\}.$$

We can decide if T is injective and surjective as follows:

Th^{m}/ If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear transformation then

(1.) T is injective iff $\text{Null}[T] = \{0\}$.

(2.) T is surjective iff $\text{Col}[T] = \mathbb{R}^m$

(3.) T is <u>invertible</u> iff $n = m$ and either (1.) or (2.) holds.

In other words, T is injective iff $\text{nullity}([T]) = 0$ and T is surjective iff $\text{rank}([T]) = m$. Then the rank-nullity Th^{m} gives that $\text{rank}[T] + \text{nullity}[T] = n$ so the only way for T to be both surjective and injective is that $m = n$ and $\text{rank}[T] = n$, $\text{nullity}[T] = 0$.

$\boxed{E1}$ $T(x,y,z) = (x+y, y+z) = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{[T]} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $T : \mathbb{R}^3 \to \mathbb{R}^2$

this map is surjective since $\text{rank}[T] = 2$ and we see $\text{Col}[T] = \mathbb{R}^2$

However $\text{nullity}[T] = 3 - \text{rank}[T] = 3 - 2 = 1 \neq 0 \therefore T$ not 1-to-1.

E2 $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $[T] = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$ has $\text{rank}[T] = 2$ whereas $\text{nullity}([T]) = 0$ thus T is one-to-one.

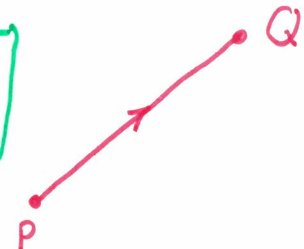
E3 $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(x, y) = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ has $n = m = 2$, $\text{rank}[T] = 2$ then $\text{nullity}[T] = 0$. T is both injective and surjective. Moreover,

$$[T]^{-1} = \frac{1}{5-4}\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

then $T^{-1}(a, b) = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$ gives formula for T^{-1}!

To understand the behavior of a linear transformation it is helpful to analyze how T maps a line-segment.

Defn/ $L(P, Q) = \{ P + t(Q - P) \mid 0 \leq t \leq 1 \}$ is the line-segment from P to Q



Thm/ If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a injective linear map and $P, Q \in \mathbb{R}^n$ then $T(L(P, Q)) = L(T(P), T(Q))$

Proof: Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ a linear map,

$$\begin{aligned} T(L(P, Q)) &= \{ T(x) \mid x \in L(P, Q) \} \\ &= \{ T(P + t(Q - P)) \mid 0 \leq t \leq 1 \} \\ &= \{ T(P) + t(T(Q) - T(P)) \mid 0 \leq t \leq 1 \} \\ &= L(T(P), T(Q)). \end{aligned}$$

Remark: we probably should require $P \neq Q$ for $L(P, Q)$ to form a line-segment. Notice $P \neq Q \Rightarrow T(P) \neq T(Q)$ provided T is injective.

VECTOR SPACES AND SUBSPACES

We say V is a vector space over $\mathbb{R}$ if V is a set which has an addition and scalar multiplication. Given $x, y \in V$ we need $x+y \in V$ and $c \cdot x \in V$ for each $c \in \mathbb{R}$. There are many examples, but I'll limit our discussion to three cases (and their subspaces)

1.) $\mathbb{R}^n$ (column vectors)

2.) $\mathbb{R}^{m \times n}$ (matrices)

3.) $P_n(\mathbb{R})$ (n^{th} order polynomials)

We say $W \subseteq V$ is a <u>subspace</u> of V if W is also a vector space, that is W is closed under addition and scalar multiplication.

Th^m/ W is a subspace of V if, $\underbrace{(\text{write } W \leq V)}_{\text{notation for } W \text{ subspace of } V.}$

(1.) W contains zero

(2.) if $x, y \in W$ implies $cx, x+y \in W$ for $c \in \mathbb{R}$.

E4 Let $A \in \mathbb{R}^{m \times n}$ then if $x, y \in Null(A)$ then note $A(x+y) = Ax + Ay = 0 + 0 = 0$ thus $x+y \in Null(A)$. Also, $x \in Null(A)$ has $A(cx) = cAx = c(0) = 0$ and finally, $A(0) = 0$ thus $0, cx \in Null(A)$ so by the subspace test Th^m we find <u>$Null(A) \leq \mathbb{R}^n$</u>.

Th^m/ If $S \subseteq V$ then $span(S) \leq V$ (spans are subspaces)

E5 $Col(A) = span\{col_1(A), col_2(A), \dots, col_n(A)\} \leq \mathbb{R}^m$