Lecture 12: Coordinate Maps & Bases

I intend this to be my most based lecture thus far. Let us recall $\beta = \{v_1, v_2, \dots, v_k\}$ is a <u>basis</u> for W if β is LI and $W = \text{span}(\beta)$. We defined $\dim(W) = \#(\beta) = k$. We now introduce a new concept, the coordinate map with respect to β

Def/ If $W = \text{span}\,\beta$ where $\beta = \{v_1, v_2, \dots, v_k\}$ is LI and $x = x_1v_1 + x_2v_2 + \cdots + x_kv_k \in W$ then $[x]_\beta = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k$

$\boxed{E1}$ Consider $W = \{A \in \mathbb{R}^{2\times 2} \mid A^T = A\}$ ← symmetric 2×2 real matrices

I'll show how to find basis. Begin by setting $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then apply $A^T = A$,

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \underbrace{d = d,\ a = a}_{\text{true.}} \ \& \ \underbrace{b = c}_{\text{helpful}}$$

Hence $A \in W$ has the form,

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, by the <u>calculation</u> above

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$ is <u>basis</u> for W.

It is <u>clear</u> β is LI and $\text{span}\,\beta = W$. Coordinate mapping

$$\boxed{\left[\begin{bmatrix} a & b \\ b & d \end{bmatrix}\right]_\beta = (a, b, d)}$$

$$\underbrace{[I_2]_\beta = (1, 0, 1)}_{\text{remember } I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.}$$

E2 Consider $W = \text{span}\underbrace{\{1, x, x^3, x^2, x^4\}}_{\text{LI set, call it } \beta \text{ serves as basis for } P_4(\mathbb{R})}$

The <u>ordering</u> of vectors in the basis β matters for example,

$$f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4$$
$$= 1 + 2x + 4x^3 + 3x^2 + 5x^4$$

$$\Rightarrow \quad \underline{[f(x)]_\beta = (1, 2, 4, 3, 5)} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \\ 5 \end{bmatrix}$$

In contrast, $\gamma = \{x^4, x^3, x^2, x, 1\}$ is a different basis and notice

$$f(x) = 5x^4 + 4x^3 + 3x^2 + 2x + 1$$

$$\Rightarrow \quad [f(x)]_\beta = (5, 4, 3, 2, 1) = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \in \mathbb{R}^5$$

Let us pause to appreciate the structure of the coordinate map. Notice the construction only makes sense because LI sets have the equating coefficient property. In particular for $\beta = \{V_1, V_2, \dots, V_n\}$ LI if $\exists$ constants $b_1, b_2, \dots, b_n$ and $c_1, c_2, \dots, c_n$ such that

$$b_1 V_1 + b_2 V_2 + \cdots + b_n V_n = c_1 V_1 + c_2 V_2 + \cdots + c_n V_n$$

then we find $b_1 = c_1$, $b_2 = c_2$, ..., $b_n = c_n$. This property of LI sets is needed for us to be certain that $[x]_\beta$ is uniquely defined by coefficients which appear in the formula $x = c_1 V_1 + \cdots + c_n V_n$.

Suppose $x, y \in W$ and $c \in \mathbb{R}$ where basis $\beta = \{v_1, v_2, \dots, v_n\}$ gives $x = \sum_{i=1}^{n} x_i v_i$ and $y = \sum_{i=1}^{n} y_i v_i$

then notice

$$cx + y = c\left(\sum_{i=1}^{n} x_i v_i\right) + \left(\sum_{i=1}^{n} y_i v_i\right)$$

$$= \sum_{i=1}^{n} (cx_i) v_i + \sum_{i=1}^{n} y_i v_i$$

$$= \sum_{i=1}^{n} (cx_i + y_i) v_i$$

Therefore,

$$[x]_\beta = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad [y]_\beta = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \& \quad [cx+y]_\beta = \begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \\ \vdots \\ cx_n + y_n \end{bmatrix}$$

Observe, $[cx+y]_\beta = c[x]_\beta + [y]_\beta$

Defn/ Given basis β for V we define $\Phi_\beta : V \to \mathbb{R}^n$ via $\Phi_\beta(x) = [x]_\beta$ for each $x \in V$. Once again we call Φ_β the β-coordinate map

Thm/ The β-coordinate map on $V = \text{span}(\beta)$ where $\beta = \{v_1, v_2, \dots, v_n\}$ is a <u>linear transformation</u> from V to $\mathbb{R}^n$

Proof: see calculations at top of this page. //

For subspaces of matrices or polynomials we pretty much just have to work out the algebra one example at a time. However, for subspaces made of column vectors there are slick tricks we can utilize to leverage past calculational strategeries.

E3 Let $\beta = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \end{bmatrix} \right\}$ find coordinates of $v = (6,7)$ in the β-coordinate system.

We wish to find $[v]_\beta = (c_1, c_2)$ where

$$\underbrace{v = \begin{bmatrix} 6 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 7 \end{bmatrix}}_{\text{this we know how to solve!}}$$

$$\begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$ multiply by 2x2 inverse!

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{7-12} \begin{bmatrix} 7 & -4 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} = \frac{-1}{5} \begin{bmatrix} 14 \\ -11 \end{bmatrix} = \begin{bmatrix} -14/5 \\ 11/5 \end{bmatrix}$$

$$\therefore \boxed{[v]_\beta = \Phi_\beta(v) = \begin{bmatrix} -14/5 \\ 11/5 \end{bmatrix}}$$

Now, suppose

$$(x,y) = \bar{x} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \bar{y} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

then $\Phi_\beta(x,y) = (\bar{x}, \bar{y}) = \frac{-1}{5} \begin{bmatrix} 7 & -4 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$\therefore \boxed{\Phi_\beta(x,y) = (\bar{x}, \bar{y}) = \left(\frac{4y - 7x}{5}, \frac{3x - y}{5} \right)}$$

change of coordinates to non-standard $\bar{x}, \bar{y}$

Let's generalize $\boxed{E3}$ to an arbitrary n-dim'l subspace of column vectors with basis $\bar{\beta} = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k\}$ (LI) with span $\beta = W$

then if $x = \bar{x}_1 \bar{v}_1 + \bar{x}_2 \bar{v}_2 + \cdots + \bar{x}_k \bar{v}_k$ we calculate

$$x = \underbrace{[\bar{v}_1 | \bar{v}_2 | \cdots | \bar{v}_k]}_{[\bar{\beta}]} \underbrace{\begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_k \end{bmatrix}}_{[x]_\beta}$$

<u>Proposition</u>: Given $W = \text{span}\,\bar{\beta}$ where $\bar{\beta} = \{\bar{v}_1, \dots, \bar{v}_k\}$ and $[\bar{\beta}] = [\bar{v}_1 | \bar{v}_2 | \cdots | \bar{v}_k]$ we have a unique $[x]_{\bar{\beta}} \in \mathbb{R}^k$ for each $x \in W$ and

$$x = [\bar{\beta}][x]_{\bar{\beta}}$$

(Here I'm allowing for $W \subseteq \mathbb{R}^n$ where $n \geq k$)

<u>Corollary</u>: Given basis $\bar{\beta} = \{\bar{v}_1, \dots, \bar{v}_n\}$ for $\mathbb{R}^n$ then $\Phi_{\bar{\beta}}(x) = [\bar{\beta}]^{-1} x$ where $[\bar{\beta}] = [\bar{v}_1 | \bar{v}_2 | \cdots | \bar{v}_n]$

<u>Proof</u>: if $x = [\bar{\beta}][x]_{\bar{\beta}}$ then $\bar{\beta}$ LI gives $[\bar{\beta}]^{-1}$ exists

hence $[\bar{\beta}]^{-1} x = [\bar{\beta}]^{-1}[\bar{\beta}][x]_{\bar{\beta}} = [x]_{\bar{\beta}}$

$$\therefore \Phi_{\bar{\beta}}(x) = [x]_{\bar{\beta}} = [\bar{\beta}]^{-1} x. \,//$$

E4 $W = \text{Null}\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\right)$

Let's find basis for W and calculate $\Phi_\beta(v)$ for some arbitrary $v \in W$.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$Ax = 0$

$\hookrightarrow x_1 + x_2 + x_3 + x_4 = 0$

$\underline{x_1 = -x_2 - x_3 - x_4}$.

$$x = \begin{bmatrix} -x_2 - x_3 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\bar{v}_1} + x_3 \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}_2} + x_4 \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\bar{v}_3} \quad (*)$$

$\beta = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is basis for $W = \text{Null}(A)$

$$\boxed{\Phi_\beta(x) = \Phi_\beta(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix}}$$ yep.

(no extra calculation required since $(*)$ reveals

E5 $W = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$ find coordinates of $(a,b,c) \in W$ w.r.t. $\beta = \left\{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = [\beta][(a,b,c)]_\beta = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

<u>find c_1 & c_2</u> (as functions of a, b, c)

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 2 & 0 & b \\ 2 & 1 & c \end{array}\right] \xrightarrow[r_3 - 2r_1]{r_2 - 2r_1} \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & -2 & b-2a \\ 0 & -1 & c-2a \end{array}\right] \xrightarrow{r_1 + r_3} \left[\begin{array}{cc|c} 1 & 0 & c-a \\ 0 & -2 & b-2a \\ 0 & -1 & c-2a \end{array}\right]$$

$\therefore$ $c_1 = c - a$
$c_2 = 2a - c$

$[(a,b,c)]_\beta = (c-a, 2a-c)$ where

E5 continued

$$(b-2a)-2(c-2a)$$

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 2 & 0 & b \\ 2 & 1 & c \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & c-a \\ 0 & -2 & b-2a \\ 0 & -1 & c-2a \end{array}\right] \xrightarrow{r_2-2r_3} \left[\begin{array}{cc|c} 1 & 0 & c-a \\ 0 & 0 & b+2a-2c \\ 0 & -1 & c-2a \end{array}\right]$$

$$\text{rref}\left[\begin{array}{cc|c} 1 & 1 & a \\ 2 & 0 & b \\ 2 & 1 & c \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & c-a \\ 0 & 1 & 2a-c \\ 0 & 0 & 2a+b-2c \end{array}\right]$$

We see $(a,b,c) \in W = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ only if $\underbrace{2a+b-2c=0}_{\begin{subarray}{l} b = 2c-2a \\ = 2(c-a) \end{subarray}}$

$$\Phi_\beta\big((a,b,c)\big) = (c-a,\ 2a-c)$$

Check our calculation,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (c-a)\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + (2a-c)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c-a+2a-c \\ 2(c-a) \\ 2(c-a)+2a-c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

E6 Let $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} \right\}$. Let's find the coordinates of $(a,b,c) = v$ w.r.t. β

$$v = [\beta][v]_\beta \quad \Rightarrow \quad [v]_\beta = [\beta]^{-1} v$$

$$[v]_\beta = \underbrace{\begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & -2 \\ 2 & 0 & 2 \end{bmatrix}^{-1}}_{\text{technology}} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 1 & 2 \\ -3 & 3 & 0 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\boxed{[v]_\beta = \left(\frac{a+b+2c}{6},\ \frac{-3a+3b}{6},\ \frac{-a-b+c}{6} \right)}$$

For example, $w = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ has $[w]_\beta = \left(\frac{6}{6}, \frac{6}{6}, 0\right) = (1,1,0)$.

E7 $W = \{ A \in \mathbb{R}^{4\times 4} \mid A^T = A,\ \mathrm{trace}(A) = 0 \}$

Find coordinate map for W

$$A^T = A \rightarrow \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix} = A$$

$$\mathrm{trace}(A) = 0 \Rightarrow a + e + h + j = 0$$

$$\therefore \underline{j = -a - e - h}.$$

$$A = \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & -a-e-h \end{bmatrix} \xrightarrow[\text{derived below}]{\text{using } \beta} [A]_\beta = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{bmatrix} = \Phi_\beta(A)$$

$$= a \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & & \\ & & & \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + d(E_{14} + E_{41}) +$$

$$+ e \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + f(E_{23} + E_{32}) + g(E_{24} + E_{42}) +$$

$$+ i(E_{34} + E_{43}) + h(E_{33} - E_{44})$$

$$= a(E_{11} - E_{44}) + b(E_{12} + E_{21}) + c(E_{13} + E_{31}) + d(E_{14} + E_{41}) +$$

$$+ e(E_{22} - E_{44}) + f(E_{23} + E_{32}) + g(E_{24} + E_{42}) + h(E_{33} - E_{44}) +$$

$$+ i(E_{34} + E_{43})$$

$$\boxed{\beta = \{ E_{11} - E_{44},\ E_{12} + E_{21},\ E_{13} + E_{31},\ E_{14} + E_{41},\ E_{22} - E_{44},\ E_{23} + E_{32},\ E_{24} + E_{42},\ E_{33} - E_{44},\ E_{34} + E_{43} \}}$$

BASIS FOR W, $\dim(W) = 9$