LECTURE 13: MATRIX OF LINEAR TRANSFORMATION OF VECTOR SPACES

We studied $L_A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and saw $[L_A] = A = [Ae_1|Ae_2|\cdots|Ae_n]$ forms the standard matrix. Or perhaps you'll remember we discussed $T: \mathbb{R}^n \to \mathbb{R}^m$ a linear map then $[T] = [T(e_1)|\cdots|T(e_n)]$ and $T(x) = [T]x$ for each $x \in \mathbb{R}^n$. We generalize now to the case of abstract vector spaces,

Def^{n}/ Let V and W be vector spaces over $\mathbb{R}$ then a function $T: V \to W$ which has

(1.) $T(x+y) = T(x) + T(y)$ (additive)

(2.) $T(cx) = cT(x)$ (homogeneous)

for each $x, y \in V$ and $c \in \mathbb{R}$, then T is linear transformation and we say $T \in \mathcal{L}(V, W)$.

$\boxed{E1}$ $V = W = C^\infty(\mathbb{R})$ ← smooth functions on $\mathbb{R}$

$T(f) = f' = \frac{df}{dx}$ has $T(f+g) = (f+g)' = f'+g' = T(f)+T(g)$

$T(cf) = (cf)' = cf' = cT(f)$

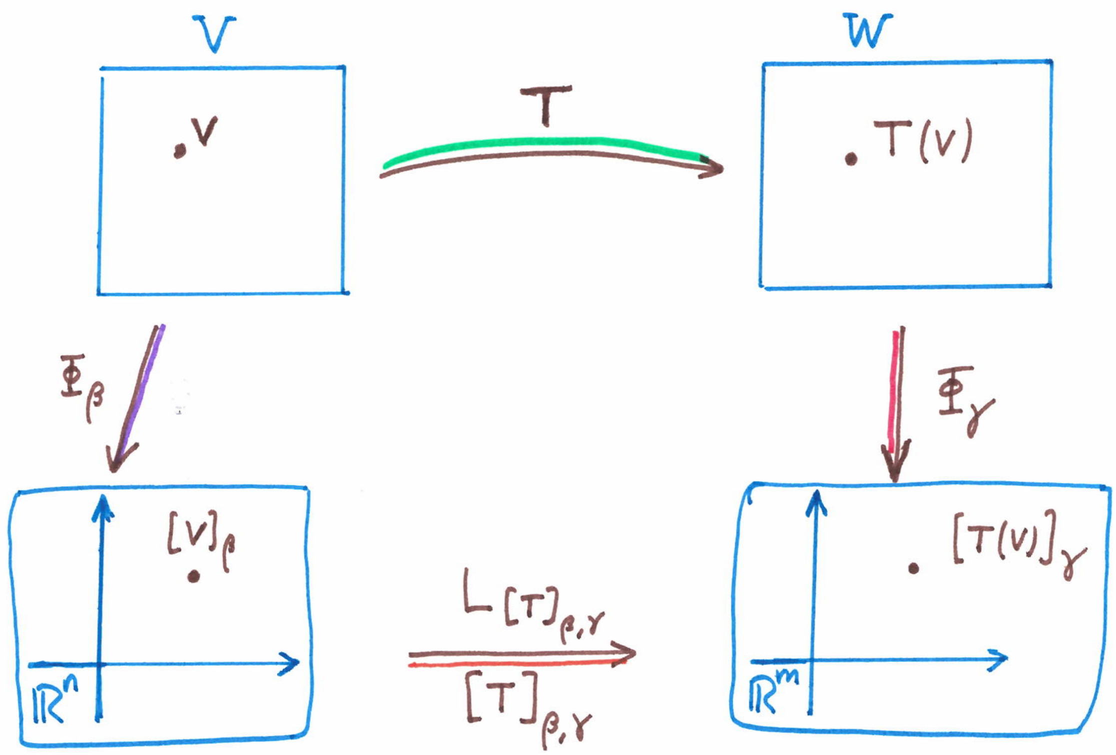
The derivative $T = d/dx$ gives an example of a linear map. Likewise $T \circ T = d^2/dx^2$ or $T^m = d^m/dx^m$ is linear on V.

QUESTION: can we write the formula for d/dx as some matrix multiplication? What is "the matrix" of d/dx?

ANSWER: well, sort of, but not for $d/dx : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ we have to narrow our focus to a finite dimensional domain $\left(\dim(C^\infty(\mathbb{R})) = \infty\right)$

MATRIX FOR $T: V \to W$ with respect to β, γ; $[T]_{\beta,\gamma}$

Let $\beta = \{V_1, V_2, \dots, V_n\}$ be basis for V and let $\gamma = \{W_1, W_2, \dots, W_m\}$ be basis for W. Also, suppose $T: V \to W$ is a linear transformation. I like to use a diagram to organize the construction,



$$V \longmapsto T(V) \longmapsto [T(V)]_\gamma$$

$$V \longmapsto [V]_\beta \longmapsto [T]_{\beta,\gamma}[V]_\beta$$

We need to define $[T]_{\beta,\gamma}$ such that $[T(V)]_\gamma = [T]_{\beta,\gamma}[\mathbf{V}]_\beta$

Now, $\Phi_\beta(V_i) = e_i$ thus $[T(V_i)]_\gamma = [T]_{\beta,\gamma} e_i$ which means $[T(V_i)]_\gamma = \text{col}_i([T]_{\beta,\gamma})$ so we define

$$\boxed{\text{Def}^{n}/ \quad [T]_{\beta,\gamma} = \big[[T(V_1)]_\gamma \,\big|\, [T(V_2)]_\gamma \,\big|\, \cdots \,\big|\, [T(V_n)]_\gamma \big]}$$

Thm/ Given bases $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_m\}$ for vector spaces V and W respective and $T : V \to W$ a linear transformation then if we define

$$[T]_{\beta,\gamma} = \big[\,[T(v_1)]_\gamma \,\big|\, [T(v_2)]_\gamma \,\big|\, \cdots \,\big|\, [T(v_n)]_\gamma\,\big]$$

then $\boxed{[T(v)]_\gamma = [T]_{\beta,\gamma}[v]_\beta}$ for all $v \in V$.

Moreover, we can use the boxed eq^{n} above to calculate $[T]_{\beta,\gamma}$.

E2 Let $T = \frac{d}{dx} : P_3(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ and lets use bases $\beta = \{1, x, x^2, x^3\}$ and $\gamma = \{1, x, x^2\}$

$$\begin{tikzcd}
\boxed{v = a + bx + cx^2 + dx^3} \arrow[r, "T"] \arrow[d, "\Phi_\beta"] & \boxed{T(v) = b + 2cx + 3dx^2} \arrow[d, "\Phi_\gamma"] \\
\boxed{[v]_\beta = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}} \arrow[r, "{[T]_{\beta,\gamma}}", "\underbrace{\qquad}_{\substack{3\times 4 \\ \text{matrix}}}"'] & \boxed{[T(v)]_\gamma = \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}}
\end{tikzcd}$$

$$\left[\qquad\qquad\right]\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix} \quad \therefore \boxed{[T]_{\beta,\gamma} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}}$$

Of course we can also calculate from definition identifying $v_1 = 1$, $v_2 = x$, $v_3 = x^2$, $v_4 = x^3$

$$\begin{aligned}
[T]_{\beta,\gamma} &= \Big[\, \big[\tfrac{d}{dx}(1)\big]_\gamma \,\Big|\, \big[\tfrac{d}{dx} x\big]_\gamma \,\Big|\, \big[\tfrac{d}{dx} x^2\big]_\gamma \,\Big|\, \big[\tfrac{d}{dx} x^3\big]_\gamma \,\Big] \\
&= \big[\, [0]_\gamma \,\big|\, [1]_\gamma \,\big|\, [2x]_\gamma \,\big|\, [3x^2]_\gamma \,\big] \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}
\end{aligned}$$

$$\boxed{\Phi_\gamma(a + bx + cx^2) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}}$$

$\boxed{E3}$ Consider $T: P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ defined by $T(f(x)) = f(x) - f'(x)$

thus $T(ax^2+bx+c) = ax^2+bx+c-(2ax+b)$
$$= ax^2+(b-2a)x+c-b$$

Let's use $\beta = \gamma = \{x^2, x, 1\}$

$$\begin{tikzcd}
\boxed{V = ax^2+bx+c} \arrow[r, "T"] \arrow[d, "\Phi_\beta"] & \boxed{T(V) = ax^2+(b-2a)x+c-b} \arrow[d, "\Phi_\beta"] \\
{[V]_\beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix}} \arrow[r, "\text{multiply}", "\text{by } {[T]_{\beta,\beta}}"'] & \begin{bmatrix} a \\ b-2a \\ c-b \end{bmatrix} = [T(V)]_\beta
\end{tikzcd}$$

$$\underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{[T]_{\beta,\beta}} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{[V]_\beta} = \underbrace{\begin{bmatrix} a \\ b-2a \\ c-b \end{bmatrix}}_{[T(V)]_\beta} \Rightarrow \boxed{[T]_{\beta,\beta} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}$$

I'll calculate directly from def$^{\underline{n}}$ as well,

$$[T]_{\beta,\beta} = \left[[T(x^2)]_\beta \mid [T(x)]_\beta \mid [T(1)]_\beta \right]$$
$$= \left[[x^2-2x]_\beta \mid [x-1]_\beta \mid [1]_\beta \right]$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

<u>Remark</u>: the notation $[T]_{\beta,\beta}$ is sometimes shortened to $[T]_\beta$, I don't do it, I write β,β so that it is consistent with my necessary use of $[T]_{\beta,\gamma}$

$\boxed{E4}$ Consider $S : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$ def$^{=}$ by $S(A) = A^T$.
Then $S(A+B) = (A+B)^T = A^T + B^T = S(A) + S(B)$ and
$S(cA) = (cA)^T = cA^T = c\,S(A)$, hence S is linear map.

$$\begin{tikzcd}
\boxed{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}} \arrow[rr, bend left, "S"] \arrow[d, bend right, "\Phi_\beta"'] & & \boxed{S(A) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}} \arrow[d, bend left, "\Phi_\beta"] \\
\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \arrow[rr, "L_{[S]_{\beta,\beta}}"] & & \begin{bmatrix} a \\ c \\ b \\ d \end{bmatrix}
\end{tikzcd}
\quad \therefore \quad \boxed{[S]_{\beta,\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Let's look at a different choice of basis, I'll motivate this choice with one of my favorite calculations

$$A = \tfrac{1}{2}(A + A^T) + \tfrac{1}{2}(A - A^T)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix}$$

$$= a\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{V_1} + (b+c)\underbrace{\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{V_2} + d\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{V_3} + (b-c)\underbrace{\frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{V_4}$$

$$\gamma = \{V_1, V_2, V_3, V_4\}$$

$$\Phi_\gamma(A) = \Phi_\gamma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a, b+c, d, b-c) = [A]_\gamma$$

Notice $V_1^T = V_1$, $V_2^T = V_2$, $V_3^T = V_3$ whereas $V_4^T = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -V_4$ thus,

$$[S]_{\gamma,\gamma} = \big[\, [S(V_1)]_\gamma \,\big|\, [S(V_2)]_\gamma \,\big|\, [S(V_3)]_\gamma \,\big|\, [S(V_4)]_\gamma \,\big]$$

$$= \big[\, [V_1]_\gamma \,\big|\, [V_2]_\gamma \,\big|\, [V_3]_\gamma \,\big|\, [-V_4]_\gamma \,\big] = \boxed{\underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right]}_{[S]_{\gamma,\gamma}}}$$

MATRIX FOR $T: \mathbb{R}^n \to \mathbb{R}^m$ w.r.t. non-standard bas

Let $\beta = \{v_1, v_2, \dots, v_n\}$ be basis for $\mathbb{R}^n$ then also let $\gamma = \{w_1, w_2, \dots, w_m\}$ be basis for $\mathbb{R}^m$ recall that we have formulas for Φ_β and Φ_γ given by

$$\underbrace{\Phi_\beta(x) = [\beta]^{-1}x}_{\text{for } x \in \mathbb{R}^n} \quad \& \quad \underbrace{\Phi_\gamma(y) = [\gamma]^{-1}y}_{\text{for } y \in \mathbb{R}^m}$$

Suppose $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping. Then,

$$\begin{aligned}
[T]_{\beta,\gamma} &= \big[[T(v_1)]_\gamma \,|\, [T(v_2)]_\gamma \,|\, \cdots \,|\, [T(v_n)]_\gamma \big] \\
&= \big[[\gamma]^{-1}T(v_1) \,|\, [\gamma]^{-1}T(v_2) \,|\, \cdots \,|\, [\gamma]^{-1}T(v_n) \big] \\
&= [\gamma]^{-1} \big[T(v_1) \,|\, T(v_2) \,|\, \cdots \,|\, T(v_n) \big] \\
&= [\gamma]^{-1} \big[[T]v_1 \,|\, [T]v_2 \,|\, \cdots \,|\, [T]v_n \big] \\
&= [\gamma]^{-1} [T] [v_1 | v_2 | \cdots | v_n] \\
&= [\gamma]^{-1} [T] [\beta]
\end{aligned}$$

Therefore,

> **Th^m/** Given linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ if β is basis for $\mathbb{R}^n$ and γ is basis for $\mathbb{R}^m$ then $[T]_{\beta,\gamma} = [\gamma]^{-1}[T][\beta]$. Moreover, if $S: \mathbb{R}^n \to \mathbb{R}^n$ then $[S]_{\beta,\beta} = [\beta]^{-1}[S][\beta]$

<u>Remark</u>: the formulas above hold for linear maps of <u>column</u> <u>vectors</u>. Otherwise, we need a different approach. (see E2, E3, E4)

$\boxed{E5}$ Let $T(x,y,z) = \underbrace{\frac{1}{38}\begin{bmatrix} 780 & -246 & -20 \\ 1016 & -238 & -28 \\ 1758 & -522 & -10 \end{bmatrix}}_{[T]}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Consider $\beta = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -9 \end{bmatrix} \right\}$ we showed in LECTURE 8 that $[\beta]^{-1} = \frac{1}{38}\begin{bmatrix} 38 & 0 & 0 \\ 39 & -18 & -1 \\ 2 & 2 & -2 \end{bmatrix}$ thus calculate,

$$
\begin{aligned}
[T]_{\beta,\beta} &= [\beta]^{-1}[T][\beta] \\
&= \frac{1}{(38)^2}\begin{bmatrix} 38 & 0 & 0 \\ 39 & -18 & -1 \\ 2 & 2 & -2 \end{bmatrix}\begin{bmatrix} 780 & -246 & -20 \\ 1016 & -238 & -28 \\ 1758 & -522 & -10 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 7 \\ 3 & 4 & -9 \end{bmatrix} \\
&= \frac{38}{(38)^2}\begin{bmatrix} 38 & 0 & 0 \\ 39 & -18 & -1 \\ 2 & 2 & -2 \end{bmatrix}\begin{bmatrix} 6 & 26 & 21 \\ 12 & 38 & 43 \\ 18 & 64 & 45 \end{bmatrix} \\
&= \frac{1}{38}\begin{bmatrix} 228 & 988 & 798 \\ 0 & 266 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \boxed{\begin{bmatrix} 6 & 26 & 21 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [T]_{\beta,\beta}}
\end{aligned}
$$

not quite diagonal.

Another choice of basis which is better is $\gamma = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 13 \\ 20 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 99 \end{bmatrix} \right\}$ then if we calculate $[T]_{\gamma\gamma}$ we'll find,

$$
[T]_{\gamma\gamma} = [\gamma]^{-1}[T][\gamma] = \underbrace{\boxed{\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [T]_{\gamma\gamma}}}_{\text{diagonalization of } [T].}
$$