LECTURE 14: COORDINATE CHANGE FOR VECTORS

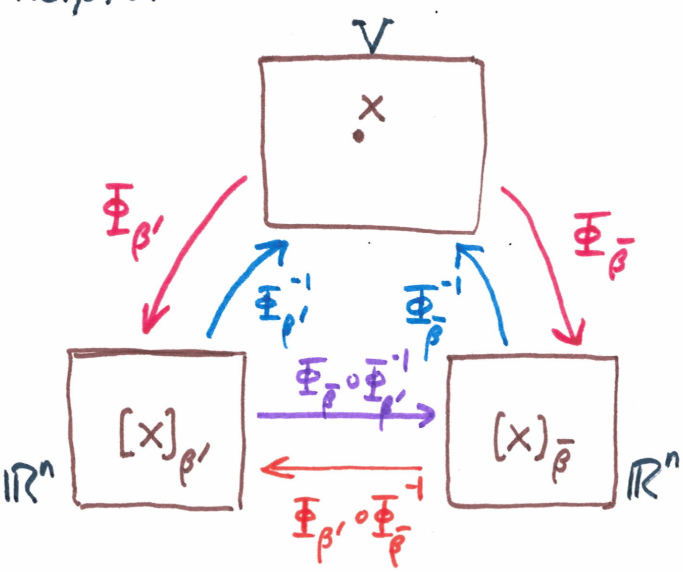
Given a vector space V with basis $\beta' = \{v_1', v_2', \dots, v_n'\}$ we may wish to also study another basis $\bar{\beta} = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}$. In principle, there are infinitely many choices for β' & $\bar{\beta}$. We wish to connect $[x]_{\beta'}$ and $[x]_{\bar{\beta}}$. Let's use $[x]_{\beta'} = (x_1', x_2', \dots, x_n')$ which means that

$$\boxed{x = x_1' v_1' + x_2' v_2' + \cdots + x_n' v_n' = \sum_{i=1}^{n} x_i' v_i'}$$

Likewise $[x]_{\bar{\beta}} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ which means

$$\boxed{x = \bar{x}_1 \bar{v}_1 + \bar{x}_2 \bar{v}_2 + \cdots + \bar{x}_n \bar{v}_n = \sum_{i=1}^{n} \bar{x}_i \bar{v}_i}$$

Diagramatically we find $\Phi_{\beta'}(x) = [x]_{\beta'}$ and $\Phi_{\bar{\beta}}(x) = [x]_{\bar{\beta}}$ a helpful notation



Thm/ with the notations given above,

$$[x]_{\bar{\beta}} = [\Phi_{\bar{\beta}} \circ \Phi_{\beta'}^{-1}][x]_{\beta'}$$

Moreover,

$$[x]_{\beta'} = [\Phi_{\beta'} \circ \Phi_{\bar{\beta}}^{-1}][x]_{\bar{\beta}}$$

Proof: $\left(\Phi_{\bar{\beta}} \circ \Phi_{\beta'}^{-1}\right)([x]_{\beta'}) = \Phi_{\bar{\beta}}\left(\Phi_{\beta'}^{-1}\left(\Phi_{\beta'}(x)\right)\right) = \Phi_{\bar{\beta}}(x) = [x]_{\bar{\beta}}$

$\Phi_{\bar{\beta}} \circ \Phi_{\beta'}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is *linear map*

$\therefore \left(\Phi_{\bar{\beta}} \circ \Phi_{\beta'}^{-1}\right)([x]_{\beta'}) = [\Phi_{\bar{\beta}} \circ \Phi_{\beta'}^{-1}][x]_{\beta'} = [x]_{\bar{\beta}}$ //

$\boxed{E1}$ Consider $V = P_3(\mathbb{R})$ with bases $\beta' = \{1, x, x^2, x^3\}$ and $\bar{\beta} = \{x^3, x^2, x, 1\}$. Let's work out how to change coordinates for $f(x) = a + bx + cx^2 + dx^3 \in P_3(\mathbb{R})$.

$$\boxed{\begin{aligned} &\text{Th}^{m}/ \quad [f(x)]_{\bar{\beta}} = [\Phi_{\bar{\beta}} \circ \Phi_{\beta'}^{-1}][f(x)]_{\beta'} \\ &\qquad\quad [f(x)]_{\beta'} = [\Phi_{\beta'} \circ \Phi_{\bar{\beta}}^{-1}][f(x)]_{\bar{\beta}} \end{aligned}}$$

$$\begin{aligned} [\Phi_{\beta'} \circ \Phi_{\bar{\beta}}^{-1}] &= [\Phi_{\beta'}(\Phi_{\bar{\beta}}^{-1}(e_1)) \,|\, \Phi_{\beta'}(\Phi_{\bar{\beta}}^{-1}(e_2)) \,|\, \Phi_{\beta'}(\Phi_{\bar{\beta}}^{-1}(e_3)) \,|\, \Phi_{\beta'}(\Phi_{\bar{\beta}}^{-1}(e_4))] \\ &= [\Phi_{\beta'}(x^3) \,|\, \Phi_{\beta'}(x^2) \,|\, \Phi_{\beta'}(x) \,|\, \Phi_{\beta'}(1)] \\ &= [e_4 | e_3 | e_2 | e_1] \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Notice $f(x) = a + bx + cx^2 + dx^3 = dx^3 + cx^2 + bx + a$

thus $[f(x)]_{\beta'} = (a, b, c, d)$ whereas $[f(x)]_{\bar{\beta}} = (d, c, b, a)$

and naturally,

$$[\Phi_{\beta'} \circ \Phi_{\bar{\beta}}^{-1}][f(x)]_{\bar{\beta}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{[\Phi_{\beta'} \circ \Phi_{\bar{\beta}}^{-1}]} \underbrace{\begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix}}_{[f(x)]_{\bar{\beta}}} = \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{[f(x)]_{\beta'}}$$

$$\boxed{\begin{aligned} &\text{Th}^{m}/ \text{ Let } \beta' = \{v_1', \dots, v_n'\} \text{ and } \bar{\beta} = \{\bar{v}_1, \dots, \bar{v}_n\} \text{ then} \\ &\quad [\Phi_{\beta'} \circ \Phi_{\bar{\beta}}^{-1}] = \underbrace{[\,[\bar{v}_1]_{\beta'} \,|\, [\bar{v}_2]_{\beta'} \,|\, \cdots \,|\, [\bar{v}_n]_{\beta'}\,]}_{\substack{\text{coordinate change matrix formed by} \\ \text{coordinate vector of } \bar{\beta} \text{ basis with respect} \\ \text{to the } \beta' \text{ basis.}}} \end{aligned}}$$

E2 Consider $V = \mathbb{R}^{2\times 2}$ with $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ and $\bar{\beta} = \{E_{11}, E_{22}, E_{12}+E_{21}, E_{12}-E_{21}\}$. Let's use notation $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ then

$$A = a\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_{11}} + b\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_{12}} + c\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_{21}} + d\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_{22}} \therefore \underline{[A]_\beta = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}.$$

Next, adapting our coordinate change theorem to the current notational choices gives,

$$[A]_{\bar{\beta}} = [\Phi_{\bar{\beta}} \circ \Phi_\beta^{-1}][A]_\beta$$

Let's try out the Th$^{\underline{m}}$ on the last page to calculate $[\Phi_{\bar{\beta}} \circ \Phi_\beta^{-1}]$,

$$[\Phi_{\bar{\beta}} \circ \Phi_\beta^{-1}] = \big[\, [E_{11}]_{\bar{\beta}} \,\big|\, [E_{12}]_{\bar{\beta}} \,\big|\, [E_{21}]_{\bar{\beta}} \,\big|\, [E_{22}]_{\bar{\beta}} \,\big]$$

$$= \left[\begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix} \,\middle|\, \left[\frac{E_{12}+E_{21}}{2} + \frac{E_{12}-E_{21}}{2} \right]_{\bar{\beta}} \,\middle|\, \left[\frac{E_{21}+E_{12}}{2} + \frac{E_{21}-E_{12}}{2} \right]_{\bar{\beta}} \,\middle|\, \begin{matrix} 0 \\ 1 \\ 0 \\ 0 \end{matrix} \right]$$

$$= \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & -1/2 & 0 \end{array} \right]$$

generally, we'd have to do algebra here, this is a sneaky trick.

Consider then,

$$[\Phi_{\bar{\beta}} \circ \Phi_\beta^{-1}][A]_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ d \\ \frac{1}{2}(b+c) \\ \frac{1}{2}(b-c) \end{bmatrix} = [A]_{\bar{\beta}}$$

Let's check my calculation, recall $\bar{\beta} = \{E_{11}, E_{22}, E_{12}+E_{21}, E_{12}-E_{21}\}$

$$aE_{11} + dE_{22} + \underbrace{\tfrac{1}{2}(b+c)(E_{12}+E_{21}) + \tfrac{1}{2}(b-c)(E_{12}-E_{21})}_{bE_{12} + cE_{21}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A \checkmark$$

Remark: in the abstract context it's probably easier for many examples to skip the construction of $[\Phi_{\bar{\beta}} \circ \Phi_{\beta'}^{-1}]$ and just directly calculate both $[x]_{\bar{\beta}}$ and $[x]_{\beta'}$. Our focus in this course is column vector. We have much nicer formulas in this case.

Consider $\mathbb{R}^n$ with basis $\beta' = \{v_1', v_2', \dots, v_n'\}$ and also basis $\bar{\beta} = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}$ then for any $x \in \mathbb{R}^n$,

$$x = x_1' v_1' + x_2' v_2' + \cdots + x_n' v_n'$$

$$= [v_1' | v_2' | \cdots | v_n'] \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$$

$$= [\beta'][x]_{\beta'}$$

Likewise,

$$x = \bar{x}_1 \bar{v}_1 + \cdots + \bar{x}_n \bar{v}_n = [\bar{\beta}][x]_{\bar{\beta}}$$

Therefore, we can calculate how $[x]_{\beta'}$ and $[x]_{\bar{\beta}}$ are related by solving $x = x$,

$$[\bar{\beta}][x]_{\bar{\beta}} = [\beta'][x]_{\beta'} \Rightarrow \boxed{[x]_{\bar{\beta}} = [\bar{\beta}]^{-1}[\beta'][x]_{\beta'}}$$

Remark: when $\Phi_{\beta'} : \mathbb{R}^n \to \mathbb{R}^n$ and $\Phi_{\bar{\beta}} : \mathbb{R}^n \to \mathbb{R}^n$ then we can simplify the change of basis matrix via our usual calculus of standard matrices,

$$[\Phi_{\bar{\beta}} \circ \Phi_{\beta'}^{-1}] = [\Phi_{\bar{\beta}}][\Phi_{\beta'}^{-1}] = [\bar{\beta}]^{-1}\left([\beta']^{-1}\right)^{-1} = [\bar{\beta}]^{-1}[\beta'].$$

$\boxed{E3}$ Consider $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Let's consider bases $\bar{\beta} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ and $\beta' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$$[v]_{\bar{\beta}} = [\bar{\beta}]^{-1} v = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} v = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \underline{\begin{bmatrix} 4 \\ 3 \end{bmatrix} = [v]_{\bar{\beta}}} \quad \text{I}$$

$$[v]_{\beta'} = [\beta']^{-1} v = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} v = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \underline{\begin{bmatrix} 7/2 \\ 1/2 \end{bmatrix} = [v]_{\beta'}} \quad \text{II}$$

$$[\bar{\beta}]^{-1} [\beta'] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\underline{[v]_{\bar{\beta}} = [\bar{\beta}]^{-1} [\beta'] [v]_{\beta'} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 7/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}} \quad \text{III}$$

We see I, II and III verify our change of coordinate formula for column vectors.

$\boxed{E4}$ Suppose $v \in \mathbb{R}^3$ has $[v]_\beta = \begin{bmatrix} a+b \\ b+c \\ c+a \end{bmatrix}$ whereas $[v]_{\bar{\beta}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\bar{\beta} = \{e_3, e_2, e_1\}$. Find β.

$$[v]_{\bar{\beta}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ and } \bar{\beta} = \{e_3, e_2, e_1\} \Rightarrow v = ae_3 + be_2 + ce_1 = \begin{bmatrix} c \\ b \\ a \end{bmatrix}$$

$$\text{Then } [(c,b,a)]_\beta = \begin{bmatrix} a+b \\ b+c \\ c+a \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\therefore [\beta]^{-1} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow [\beta]^{-1} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\Rightarrow [\beta]^{-1} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow \underline{[\beta] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1}}.$$

Columns of $[\beta]$ would then be the desired β.

E5 Let $V_1 = (1,1,0,0)$, $V_2 = (1,-1,0,0)$ and $V_3 = (0,0,1,1)$ and $V_4 = (0,0,1,-1)$.
Calculate $[(x,y,z,w)]_\beta$ for $\beta = \{V_1, V_2, V_3, V_4\}$.

$$[(x,y,z,w)]_\beta = [\beta]^{-1}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} & 0 \\ \hline 0 & \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \end{array}\right]\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad \text{—} \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{-1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ \hline 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & -1/2 \end{array}\right]\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(x+y) \\ \frac{1}{2}(x-y) \\ \frac{1}{2}(z+w) \\ \frac{1}{2}(z-w) \end{bmatrix}$$

$$= \boxed{\frac{1}{2}(x+y,\ x-y,\ z+w,\ z-w)}$$

ORTHONORMAL COORDINATES ARE NICE

Given basis of vectors of length one which are pairwise perpendicular we'll find coordinate vectors are very easy to calculate.

Def^{n}/ Let $\bar{\beta} = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_m\}$ be a subset of vectors for which $\bar{v}_i \cdot \bar{v}_j = \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$ then $\bar{\beta}$ is an <u>orthonormal</u> set.

For example, $e_i \cdot e_j = \delta_{ij}$ so $\{e_1, e_2, \dots, e_n\}$ is an orthonormal set of vectors, the standard basis is an orthonormal basis. Notice $x \in \mathbb{R}^n$ has $x = (x \cdot e_1, x \cdot e_2, \dots, x \cdot e_n)$. This generalizes,

Th^{m}/ If $W = \text{span}\,\bar{\beta}$ where $\bar{\beta} = \{\bar{v}_1, \dots, \bar{v}_k\}$ is orthonormal and if $x \in W$ then $x = \sum_{i=1}^{k} (x \cdot \bar{v}_i)\bar{v}_i$,

$$[x]_{\bar{\beta}} = (x \cdot \bar{v}_1, x \cdot \bar{v}_2, \dots, x \cdot \bar{v}_k)$$

<u>Proof</u>: if $x \in W$ and $x = \sum_{i=1}^{k} \;_i \bar{v}_i$ then

$$x \cdot \bar{v}_j = \left(\sum_{i=1}^{k} \bar{x}_i \bar{v}_i\right) \cdot \bar{v}_j = \sum_{i=1}^{k} \bar{x}_i \underbrace{\bar{v}_i \cdot \bar{v}_j}_{\delta_{ij}} = \bar{x}_j$$

Therefore, $x = \sum_{i=1}^{k} (x \cdot \bar{v}_i)\bar{v}_i$ hence $[x]_{\bar{\beta}} = (x \cdot \bar{v}_1, \dots, x \cdot \bar{v}_k)$. //

$\boxed{E6}$ $W = \text{span}\,\bar{\beta}$ where $\bar{\beta} = \{\bar{v}_1, \bar{v}_2\}$ and $\bar{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\bar{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ then $(2, 0, 1) = x \in W$ and

$$[x]_{\bar{\beta}} = \begin{bmatrix} x \cdot \bar{v}_1 \\ x \cdot \bar{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{2+1}{\sqrt{3}} \\ \frac{2+0}{\sqrt{2}} \end{bmatrix} = \boxed{\begin{bmatrix} \sqrt{3} \\ \sqrt{2} \end{bmatrix}}$$

MATRIX FOR ORTHONORMAL BASIS

Given orthonormal basis $\bar{\beta} = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}$ for $\mathbb{R}^n$ we found for any $x \in \mathbb{R}^n$,

$$[x]_{\bar{\beta}} = \begin{bmatrix} x \cdot \bar{v}_1 \\ x \cdot \bar{v}_2 \\ \vdots \\ x \cdot \bar{v}_n \end{bmatrix} = \begin{bmatrix} \bar{v}_1^T x \\ \hline \bar{v}_2^T x \\ \hline \vdots \\ \hline \bar{v}_n^T x \end{bmatrix} = \begin{bmatrix} \bar{v}_1^T \\ \hline \bar{v}_2^T \\ \hline \vdots \\ \hline \bar{v}_n^T \end{bmatrix} x = [\bar{\beta}]^T x$$

Here $[\bar{\beta}]^T = [\bar{v}_1 | \bar{v}_2 | \cdots | \bar{v}_n]^T = \begin{bmatrix} \bar{v}_1^T \\ \hline \bar{v}_2^T \\ \hline \vdots \\ \hline \bar{v}_n^T \end{bmatrix}$ is why we found

But, we also know for any basis β of $\mathbb{R}^n$ we have $[x]_\beta = [\beta]^{-1} x$ thus applying this to $\bar{\beta}$ we find

$$[x]_{\bar{\beta}} = [\bar{\beta}]^{-1} x = [\bar{\beta}]^T x \quad (\text{for all } x \in \mathbb{R}^n)$$

Take $x = e_j$ to see $\text{col}_j([\bar{\beta}]^{-1}) = \text{col}_j([\bar{\beta}]^T)$ for $j = 1, 2, \dots, n$. Therefore,

$$\boxed{[\bar{\beta}]^{-1} = [\bar{\beta}]^T}$$

Such matrices have a name,

Def^n/ If $R \in \mathbb{R}^{n \times n}$ and $R^T R = I$ then we say R is an <u>orthogonal matrix</u> and that $R \in O_n(\mathbb{R})$