LECTURE 15: COORDINATE CHANGE FOR LINEAR TRANSFORMATIONS

Consider $T: V \longrightarrow W$ where V & W are vector spaces over $\mathbb{R}$ with bases $\beta = \{v_1, .., v_n\}$ and $\gamma = \{w_1, .., w_m\}$ then we derived in LECTURE 13 that

$$[T]_{\beta,\gamma} = \big[[T(v_1)]_\gamma \,|\, \cdots \,|\, [T(v_n)]_\gamma \big]$$

and I explained that this was equivalent to stating

$$[T]_{\beta,\gamma} = [\,\Phi_\gamma \circ T \circ \Phi_\beta^{-1}\,]$$

We can see the formula above from the fact that $[L_A] = A$ for $A \in \mathbb{R}^{m \times n}$ and $L_{[T]_{\beta,\gamma}} = \Phi_\gamma \circ T \circ \Phi_\beta^{-1}$

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\Phi_\beta"'] & W \arrow[d, "\Phi_\gamma"] \\
\mathbb{R}^n \arrow[r, "L_{[T]_{\beta,\gamma}}"] & \mathbb{R}^m
\end{tikzcd}$$

Ok, so what if we add another pair of bases? How will $[T]_{\beta,\gamma}$ be related to $[T]_{\bar{\beta},\bar{\gamma}}$? Algebraically, we can expect

$$[T]_{\bar{\beta},\bar{\gamma}} = [\,\Phi_{\bar{\gamma}} \circ T \circ \Phi_{\bar{\beta}}^{-1}\,]$$

We wish to relate $A = [T]_{\beta,\gamma}$ to $\bar{A} = [T]_{\bar{\beta},\bar{\gamma}}$

$$
\begin{tikzcd}
 & V \arrow[rrrr, "T"] \arrow[dl, "\Phi_\beta"'] \arrow[dr, "\Phi_{\bar{\beta}}"] & & & & W \arrow[dl, "\Phi_{\bar{\gamma}}"'] \arrow[dr, "\Phi_\gamma"] & \\
\mathbb{R}^n \arrow[rr, "\Phi_{\bar{\beta}} \circ \Phi_\beta^{-1}"] \arrow[rrrrrr, bend right, "L_A"'] & & \mathbb{R}^n \arrow[rr, "L_{\bar{A}}"] & & \mathbb{R}^m \arrow[rr, "\Phi_\gamma \circ \Phi_{\bar{\gamma}}^{-1}"] & & \mathbb{R}^m
\end{tikzcd}
$$

From the diagram I see that

$$L_A = (\Phi_\gamma \circ \Phi_{\bar{\gamma}}^{-1}) \circ L_{\bar{A}} \circ (\Phi_{\bar{\beta}} \circ \Phi_\beta^{-1})$$

But, $[L_A] = A = [T]_{\beta,\gamma}$ and $[L_{\bar{A}}] = [T]_{\bar{\beta},\bar{\gamma}}$ thus taking the standard matrix of both sides and noting $L_{\bar{A}} : \mathbb{R}^n \to \mathbb{R}^m$, $\Phi_\gamma \circ \Phi_\gamma^{-1} : \mathbb{R}^m \to \mathbb{R}^m$ and $\Phi_{\bar{\beta}} \circ \Phi_\beta^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ are all linear maps and we know $[T_1 \circ T_2] = [T_1][T_2]$ hence,

$$[L_A] = [\Phi_\gamma \circ \Phi_{\bar{\gamma}}^{-1}][L_{\bar{A}}][\Phi_{\bar{\beta}} \circ \Phi_\beta^{-1}]$$

Thm/ $[T]_{\beta,\gamma} = [\Phi_\gamma \circ \Phi_{\bar{\gamma}}^{-1}][T]_{\bar{\beta},\bar{\gamma}}[\Phi_{\bar{\beta}} \circ \Phi_\beta^{-1}]$

and $[T]_{\bar{\beta},\bar{\gamma}} = [\Phi_{\bar{\gamma}} \circ \Phi_\gamma^{-1}][T]_{\beta,\gamma}[\Phi_\beta \circ \Phi_{\bar{\beta}}^{-1}]$

SPECIAL CASE: $T: V \to V$ with $\beta = \gamma$, $\bar{\beta} = \bar{\gamma}$

In the case $T: V \to V$ we say T is a linear transformation on V. If we use the same basis for both input/output of T then the coordinate change formulas are extra nice, $\underbrace{[T]_{\bar{\beta},\bar{\beta}} \text{ and } [T]_{\beta,\beta}}$ are <u>similar</u> matrices.

Th$^{\underline{m}}$/ $[T]_{\bar{\beta},\bar{\beta}} = P^{-1}[T]_{\beta,\beta}P$ where $P = [\Phi_\beta \circ \Phi_{\bar{\beta}}^{-1}]$ and to be explicit for future reference, $\beta = \{v_1, \dots, v_n\}$ and $\bar{\beta} = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}$ gives

$$P = \big[\,[\bar{v}_1]_\beta \,|\, [\bar{v}_2]_\beta \,|\, \cdots \,|\, [\bar{v}_n]_\beta\,\big]$$

<u>Proof</u>: notice $P = [\Phi_\beta \circ \Phi_{\bar{\beta}}^{-1}]$ has that

$$P^{-1} = [(\Phi_\beta \circ \Phi_{\bar{\beta}}^{-1})^{-1}] = [\Phi_{\bar{\beta}} \circ \Phi_\beta^{-1}] \text{ then}$$

the claim $[T]_{\bar{\beta},\bar{\beta}} = P^{-1}[T]_{\beta,\beta}P$ follows immediately from the Th$^{\underline{m}}$ on pg. 2. Then,

$$\begin{aligned} P &= [\,\Phi_\beta \circ \Phi_{\bar{\beta}}^{-1}(e_1) \,|\, \cdots \,|\, \Phi_\beta \circ \Phi_{\bar{\beta}}^{-1}(e_n)\,] \\ &= [\,\Phi_\beta(\bar{v}_1) \,|\, \cdots \,|\, \Phi_\beta(\bar{v}_n)\,] \\ &= [\,[\bar{v}_1]_\beta \,|\, \cdots \,|\, [\bar{v}_n]_\beta\,]. \quad // \end{aligned}$$

Def$^{\underline{n}}$/ $A, B \in \mathbb{R}^{n \times n}$ are <u>similar</u> and we write $A \sim B$ if there exists P invertible with $B = P^{-1}AP$.

Remark: in-class we began with $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and derived the coordinate change rule via matrix calculation with the <u>standard</u> <u>matrix</u> $[T]$, that argument I provide now,

Suppose $\mathbb{R}^n$ has bases β and $\bar{\beta}$ whereas $\mathbb{R}^m$ has basis γ and $\bar{\gamma}$ then since $V = [\beta][V]_\beta$ and $W = [\gamma][W]_\gamma$ are identities which essentially <u>define</u> $[V]_\beta$ & $[W]_\gamma$ we can apply these to

$$T(x) = [T]x \;\Rightarrow\; [\gamma][T(x)]_\gamma = [T][\beta][x]_\beta$$
$$\Rightarrow\; [T(x)]_\gamma = [\gamma]^{-1}[T][\beta][x]_\beta$$

But, $[T(x)]_\gamma = [T]_{\beta,\gamma}[x]_\beta$ $\therefore$ $\boxed{[T]_{\beta,\gamma} = [\gamma]^{-1}[T][\beta]}$

Likewise, $\boxed{[T]_{\bar{\beta},\bar{\gamma}} = [\bar{\gamma}]^{-1}[T][\bar{\beta}]}$

We can solve both of these for $[T]$,

$$[T] = [\gamma][T]_{\beta,\gamma}[\beta]^{-1} = [\bar{\gamma}][T]_{\bar{\beta},\bar{\gamma}}[\bar{\beta}]^{-1}$$

$$\therefore \quad \boxed{[T]_{\bar{\beta},\bar{\gamma}} = [\bar{\gamma}]^{-1}[\gamma][T]_{\beta,\gamma}[\beta]^{-1}[\bar{\beta}]}$$

We find for $T : \mathbb{R}^n_{\beta,\bar\beta} \longrightarrow \mathbb{R}^m_{\gamma,\bar\gamma}$ that

$$\boxed{[T]_{\bar\beta,\bar\gamma} = \underline{[\bar\gamma]^{-1}[\gamma]}\,[T]_{\beta,\gamma}\,\underline{[\beta]^{-1}[\bar\beta]}}$$

This is consistent with the Thm on pg. 2 since in the case $\Phi_\beta : \mathbb{R}^n \to \mathbb{R}^n$, $\Phi_\gamma : \mathbb{R}^m \to \mathbb{R}^m$ we can simplify

$$[\Phi_{\bar\gamma} \circ \Phi_\gamma^{-1}] = [\Phi_{\bar\gamma}][\Phi_\gamma]^{-1} = [\bar\gamma]^{-1}\left([\gamma]^{-1}\right)^{-1} = \underline{[\bar\gamma]^{-1}[\gamma]}$$

$$[\Phi_\beta \circ \Phi_{\bar\beta}^{-1}] = [\Phi_\beta][\Phi_{\bar\beta}]^{-1} = [\beta]^{-1}\left([\bar\beta]^{-1}\right)^{-1} = \underline{[\beta]^{-1}[\bar\beta]}$$

Often, indeed most often, we study $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with $\beta = \gamma$ then

> **Thm/** Given $T : \mathbb{R}^n \to \mathbb{R}^n$ linear map and bases $\beta, \bar\beta$ for $\mathbb{R}^n$ we have
> $$[T]_{\bar\beta,\bar\beta} = P^{-1}[T]_{\beta,\beta}P$$
> where $P = [\beta]^{-1}[\bar\beta] = \big[\,[\bar v_1]_\beta \,|\, [\bar v_2]_\beta \,|\cdots|\, [\bar v_n]_\beta\,\big]$
> using $\bar\beta = \{\bar v_1, \bar v_2, \dots, \bar v_n\}$

$\boxed{E1}$ Suppose you're given $[T]_{\beta,\beta} = \begin{bmatrix} -32 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 100 \end{bmatrix}$ and $[T]_{\bar\beta,\bar\beta} = \begin{bmatrix} 7 & 7 & 7 \\ 7 & 7 & 7 \\ 7 & 7 & 7 \end{bmatrix}$ then this is <u>impossible</u> because $\text{trace}(P^{-1}AP) = tr(PP^{-1}A) = tr(A)$ yet $tr[T]_{\bar\beta,\bar\beta} = 21 \neq 67$.