Lecture 18: Cramer's Rule and the Inverse Formula

Let us begin with a derivation of Cramer's Rule. First notice

$$x = \det\begin{bmatrix} x & 0 & 0 \\ y & 1 & 0 \\ z & 0 & 1 \end{bmatrix} \quad \text{and} \quad y = \det\begin{bmatrix} 1 & x & 0 \\ 0 & y & 0 \\ 0 & z & 1 \end{bmatrix} \quad \text{and} \quad z = \det\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z \end{bmatrix}$$

Then for $x = (x_1, x_2, x_3, \dots, x_n) \in \mathbb{R}^n$ observe

$$\begin{array}{c} \\ \\ j^{th} \text{ row} \rightarrow \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & x_j & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} = [e_1|e_2|\cdots|x_je_j|\cdots|e_n]^T$$

Then, since $\det(A^T) = \det(A)$ we find,

$$\det\left[e_1\middle|e_2\middle|\cdots\middle|x\middle|\cdots\middle|e_n\right] = \det\underbrace{\left[e_1\middle|e_2\middle|\cdots\middle|x_je_j\middle|\cdots\middle|e_n\right]}_{\text{diagonal matrix}} = x_j$$

Suppose A is invertible and $Ax = b$ then $x = A^{-1}b$. Let $A = [A_1|A_2|\cdots|A_n]$ then $A^{-1}A = [A^{-1}A_1|A^{-1}A_2|\cdots|A^{-1}A_n] = [e_1|e_2|\cdots|e_n]$ thus

$$\begin{aligned} x_j &= \det[e_1|e_2|\cdots|x|\cdots|e_n] \\ &= \det[A^{-1}A_1|A^{-1}A_2|\cdots|A^{-1}b|\cdots|A^{-1}A_n] \\ &= \det\left(A^{-1}[A_1|A_2|\cdots|b|\cdots|A_n]\right) \\ &= \det(A^{-1})\det[A_1|A_2|\cdots|b|\cdots|A_n] \end{aligned}$$

Therefore,

Th$^{\underline{m}}$ (Cramer's Rule) If $A \in \mathbb{R}^{n\times n}$ and $\det(A) \neq 0$ then the solution to $Ax = b$ is given by

$$x_j = \frac{\det[A_1|A_2|\cdots|A_{j-1}|b|A_{j+1}|\cdots|A_n]}{\det(A)}$$

E1 Solve $\begin{pmatrix} 6x + 7y = a \\ x + y = b \end{pmatrix}$ via Cramer's Rule.

We're solving $\begin{bmatrix} 6 & 7 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ hence,

$$x = \frac{\det\begin{bmatrix} a & 7 \\ b & 1 \end{bmatrix}}{\det\begin{bmatrix} 6 & 7 \\ 1 & 1 \end{bmatrix}} = \frac{a - 7b}{-1} = 7b - a.$$

$$y = \frac{\det\begin{bmatrix} 6 & a \\ 1 & b \end{bmatrix}}{-1} = \frac{6b - a}{-1} = a - 6b$$

Thus $\boxed{(7b - a,\ a - 6b)}$ is the sol^{n}.

E2 Consider $x^2 + y^2 + z^2 + w^2 = 1$ and $xy + zw = 2$. Then we obtain

$$2x\,dx + 2y\,dy + 2z\,dz + 2w\,dw = 0$$
$$y\,dx + x\,dy + w\,dz + z\,dw = 0$$

treat dx, dy, dz, dw as variables and solve for dz & dw.

$$2z\,dz + 2w\,dw = -2x\,dx - 2y\,dy$$
$$w\,dz + z\,dw = -y\,dx - x\,dy$$

$$\underbrace{\begin{pmatrix} z & w \\ w & z \end{pmatrix}}_{A}\begin{bmatrix} dz \\ dw \end{bmatrix} = \underbrace{\begin{bmatrix} -x\,dx - y\,dy \\ -y\,dx - x\,dy \end{bmatrix}}_{b}, \quad \det(A) = z^2 - w^2$$

$$dz = \frac{\det\left[\begin{array}{c|c} -x\,dx - y\,dy & w \\ -y\,dx - x\,dy & z \end{array}\right]}{z^2 - w^2} = \frac{z(-x\,dx - y\,dy) - w(-y\,dx - x\,dy)}{z^2 - w^2}$$

$$dw = \frac{\det\left[\begin{array}{c|c} z & -x\,dx - y\,dy \\ w & -y\,dx - x\,dy \end{array}\right]}{z^2 - w^2} = \frac{z(-y\,dx - x\,dy) - w(-x\,dx - y\,dy)}{z^2 - w^2}$$

Then, I'll clean-up dz,

$$dz = \underbrace{\left(\frac{wy - zx}{z^2 - w^2}\right)}_{\left.\frac{\partial z}{\partial x}\right|_y} dx + \underbrace{\left(\frac{wx - zy}{z^2 - w^2}\right)}_{\left.\frac{\partial z}{\partial y}\right|_x} dy$$

Remark: both of these are better solved via mult. by 2x2 inverse.

$\boxed{E3}$ Solve $\begin{pmatrix} x_1 + 2x_2 - x_3 + x_4 = 1 \\ x_1 + 3x_3 = 2 \\ x_1 - x_2 + 2x_3 + 3x_4 = 3 \\ 2x_2 - x_3 + x_4 = 4 \end{pmatrix}$ for x_2.

$$\underbrace{\begin{pmatrix} 1 & 2 & -1 & 1 \\ 1 & 0 & 3 & 0 \\ 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\det(A) = -\det\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} - 3\det\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 3 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= -2(2+3) - 1(-1-6) - 1(1-4) - 3\left[1(-1-6) - 2(1) + 1(2)\right]$$

$$= -10 + 10 - 3(-7)$$

$$= \underline{21}.$$

$$\det\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 3 \\ 0 & 4 & -1 & 1 \end{pmatrix} = \det\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 4 & -1 \\ 0 & 2 & 3 & 2 \\ 0 & 4 & -1 & 1 \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & -5 & 4 \\ 0 & 0 & -17 & 5 \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & 4 & -1 \\ 0 & -5 & 4 \\ 0 & -17 & 5 \end{bmatrix}$$

$$= \det\begin{bmatrix} -5 & 4 \\ -17 & 5 \end{bmatrix}$$

$$= \underline{43}.$$

$$\therefore \boxed{x_2 = \frac{43}{21}}$$

FORMULA FOR THE INVERSE MATRIX

Suppose A is invertible with inverse $A^{-1} = [W_1 | W_2 | \cdots | W_n]$ then $AA^{-1} = [AW_1 | AW_2 | \cdots | AW_n] = [e_1 | e_2 | \cdots | e_n]$ thus $Aw_j = e_j$ for $j = 1, 2, \dots, n$. Thus by Cramer's Rule,

(i^{th} column)

$$(W_j)_i = \frac{\det[A_1 | \cdots | A_{i-1} | e_j | A_{i+1} | \cdots | A_n]}{\det(A)}$$

$$= \frac{(-1)^{i-1}}{\det(A)} \det[e_j | A_1 | \cdots | A_{i-1} | A_{i+1} | \cdots | A_n]$$

$$= \frac{(-1)^{i-1}(-1)^{1+j}}{\det(A)} \det(\cancel{A}_{ji})$$

expanding down the 1^{st} column e_j is only nonzero for j^{th} row.

$$= \frac{(-1)^{i+j}\det(\cancel{A}_{ji})}{\det(A)}$$

Now, $\mathrm{col}_j(A^{-1}) = W_j$ and so $(W_j)_i = (A^{-1})_{ij}$

$$\boxed{(A^{-1})_{ij} = \frac{1}{\det(A)} (-1)^{i+j} \det(\cancel{A}_{ji})}$$

$$\boxed{\text{Def}^{\underline{n}}\ (\mathrm{adj}(A))_{ij} = (-1)^{i+j}\det(\cancel{A}_{ji})}$$

← classical adjoint of A

Then we have $\underline{A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)}$.

$\boxed{\text{E4}}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \cancel{A}_{11} = d,\ \cancel{A}_{12} = c,\ \cancel{A}_{21} = b,\ \cancel{A}_{22} = a$$

$$\mathrm{adj}(A) = \begin{bmatrix} \cancel{A}_{11} & -\cancel{A}_{21} \\ -\cancel{A}_{12} & \cancel{A}_{22} \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\therefore \boxed{A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

Th^{m}/ Let $A \in \mathbb{R}^{n \times n}$ with $\det(A) \neq 0$ then
$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$
where we define the adjoint of A via
$$(\text{adj}(A))_{ij} = (-1)^{i+j} \det(A_{ji})$$

I don't use this formula much, but it is very nice to know about for certain theoretical arguments, for example $A \longmapsto A^{-1}$ is a continuous function of A since the formula above is a rational function of the components of A.

Just for fun, let's use it,

sorry, I try not to use this notation,
$$|A_{31}| = \det(A_{31})$$

$\boxed{\text{E5}}$
$$A = \begin{bmatrix} a & 0 & b \\ 0 & g & 0 \\ c & 0 & d \end{bmatrix}$$

$$\det(A_{11}) = \det\begin{bmatrix} g & 0 \\ 0 & d \end{bmatrix} = gd \qquad |A_{21}| = \det\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = 0 \qquad |A_{31}| = \det\begin{bmatrix} 0 & b \\ g & 0 \end{bmatrix} = -bg$$

$$\det(A_{12}) = \det\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} = 0 \qquad |A_{22}| = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \qquad |A_{32}| = \det\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = 0$$

$$\det(A_{13}) = \det\begin{bmatrix} 0 & g \\ c & 0 \end{bmatrix} = -gc \qquad |A_{23}| = \det\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = 0 \qquad |A_{33}| = \det\begin{bmatrix} a & 0 \\ 0 & g \end{bmatrix} = ag$$

$$\text{adj}(A) = \left[(-1)^{i+j} \det(A_{ji})\right] = \begin{bmatrix} gd & 0 & -bg \\ 0 & ad-bc & 0 \\ -gc & 0 & ag \end{bmatrix}$$

$$\det(A) = a \det\begin{bmatrix} g & 0 \\ 0 & d \end{bmatrix} - 0 + b \det\begin{bmatrix} 0 & g \\ c & 0 \end{bmatrix} = agd - bcg = \underline{g(ad-bc)}.$$

$$\boxed{A^{-1} = \frac{1}{g(ad-bc)} \begin{bmatrix} gd & 0 & -bg \\ 0 & ad-bc & 0 \\ -gc & 0 & ag \end{bmatrix}}$$