Lecture 19: Euclidean Geometry & Orthogonality

Concepts of distance and angle in $\mathbb{R}^n$ can be defined in terms of the dot-product of vectors.

Def^n/ $\vec{v} \cdot \vec{w} = v^T w = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$ for $v, w \in \mathbb{R}^n$.

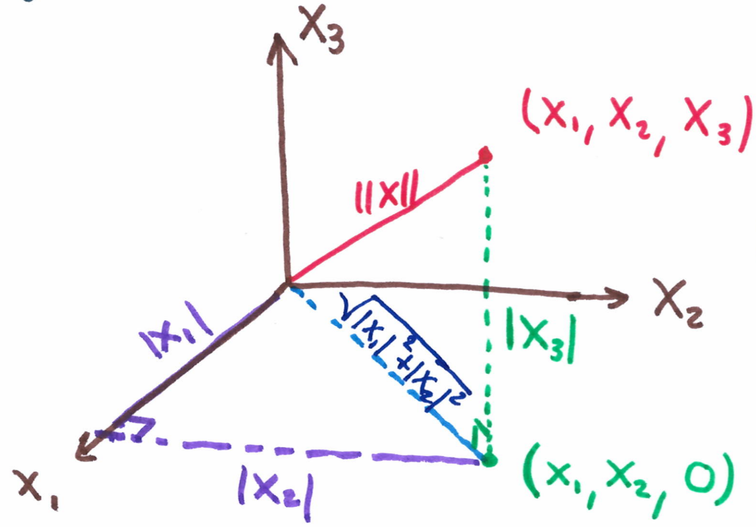
Then $\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{v \cdot v}$ defines the length of v. If $v, w \in \mathbb{R}^n$ and both v and w are nonzero then the angle between v and w is denoted $\angle(v, w)$ and is defined by

$$\angle(v, w) = \cos^{-1}\left(\frac{v \cdot w}{\|v\| \, \|w\|}\right)$$

$\mathbb{R}^n$ paired with the above concept of vector length and angle defines Euclidean Space $\mathbb{R}^n$.

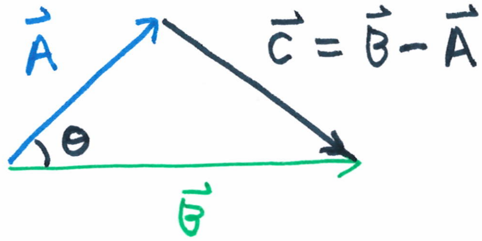
Note also, $d(P, Q) = \| Q - P \|$ gives the distance from P to Q.

The formulas above are reasonable for both algebraic and geometric reasons,



$$\begin{aligned} \|x\|^2 &= \left|\sqrt{|x_1|^2 + |x_2|^2}\right|^2 + |x_3|^2 \\ &= x_1^2 + x_2^2 + x_3^2 \\ &= x \cdot x \\ &= x^T x. \end{aligned}$$

Cauchy Schwarz Inequality & Law of Cosines



(Defn/ If use vector notation $\vec{A}$ then $A = \|\vec{A}\|$ is nice)
(Physics, $\vec{A} = A\hat{A}$)

$$\|\vec{C}\|^2 = (\vec{B}-\vec{A})\cdot(\vec{B}-\vec{A}) = \vec{B}\cdot\vec{B} - 2\vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{A}$$

$$C^2 = A^2 + B^2 - 2\vec{A}\cdot\vec{B} = A^2+B^2-2AB\cos\theta$$

(Law of Cosines)

$$\boxed{\text{Thm/ } \vec{A}\cdot\vec{B} = \|\vec{A}\|\|\vec{B}\|\cos\theta}$$

This means we can solve for $\theta = \cos^{-1}\left(\dfrac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|\|\vec{B}\|}\right)$.
Then, since $|\cos\theta| \leq 1$ for all θ we find $|\vec{A}\cdot\vec{B}| = \big|\|\vec{A}\|\|\vec{B}\|\cos\theta\big| \leq \big|\|\vec{A}\|\|\vec{B}\|\big| = \|\vec{A}\|\|\vec{B}\|$.

$$\boxed{\text{Thm/ } |v\cdot w| \leq \|v\|\|w\| \text{ for all } v,w\in\mathbb{R}^n}$$

Defn/ If $S = \{v_1, v_2, \dots, v_k\} \subseteq \mathbb{R}^n$ and $v_i\cdot v_j = 0$ for $i\neq j$ then S is an <u>orthogonal set</u>. If each vector in S has length one then S is <u>orthonormal</u>. Equivalently, S orthonormal iff $v_i\cdot v_j = \delta_{ij} = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$

We say $v \perp w$ whenever $v\cdot w = 0$, such vectors are said to be orthogonal.

E1 $\{e_1, e_2, \dots, e_n\}$ has $e_i\cdot e_j = \delta_{ij}$ the standard basis is orthonormal.

Orthogonal vectors have very nice properties, we saw before that when $\beta = \{v_1, \dots, v_n\}$ is orthonormal basis for $\mathbb{R}^n$ then $[x]_\beta = (x \cdot v_1, x \cdot v_2, \dots, x \cdot v_n)$ and $[\beta]^{-1} = [\beta]^T$. Let's prove it again. It's important,

Th^{m}/ If $\beta = \{v_1, v_2, \dots, v_n\}$ is an orthonormal basis for $\mathbb{R}^n$ then $x = \sum_{i=1}^{n} (x \cdot v_i) v_i$ hence

$$[x]_\beta = (x \cdot v_1, x \cdot v_2, \dots, x \cdot v_n) = [\beta]^T x.$$

Indeed, $[\beta]^{-1} = [\beta]^T$

<u>Proof</u>: suppose $v_i \cdot v_j = \delta_{ij}$ for $1 \leq i, j \leq n$. Let $x \in \mathbb{R}^n$ then since $\beta = \{v_1, \dots, v_n\}$ is basis we know $x = \sum_{i=1}^{n} c_i v_i$ for some constants $c_1, \dots, c_n$,

$$\begin{aligned} x \cdot v_j &= \left(\sum_{i=1}^{n} c_i v_i\right) \cdot v_j \\ &= \sum_{i=1}^{n} c_i v_i \cdot v_j \\ &= \sum_{i=1}^{n} c_i \delta_{ij} \\ &= c_j \qquad \therefore \quad \underline{x = \sum_{i=1}^{n} (x \cdot v_i) v_i}. \end{aligned}$$

Then calculate,

$$[x]_\beta = \begin{bmatrix} v_1 \cdot x \\ v_2 \cdot x \\ \vdots \\ v_n \cdot x \end{bmatrix} = \begin{bmatrix} v_1^T x \\ \hline v_2^T x \\ \hline \vdots \\ \hline v_n^T x \end{bmatrix} = \begin{bmatrix} v_1^T \\ \hline v_2^T \\ \hline \vdots \\ \hline v_n^T \end{bmatrix} x = (\beta)^T x = [\beta]^{-1} x$$

$\therefore [\beta]^T = [\beta]^{-1}$ since the above holds $\forall x \in \mathbb{R}^n$. //

Thm/ Let $x, y \in \mathbb{R}^n$ with $x \cdot y = 0$ then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

Proof: Suppose $x, y \in \mathbb{R}^n$ with $x \cdot y = 0$

$$\begin{aligned} \|x+y\|^2 &= (x+y)\cdot(x+y) \\ &= x\cdot x + x\cdot y + y \cdot x + y \cdot y \\ &= \|x\|^2 + \|y\|^2. // \end{aligned}$$

Corollary: If $S = \{x_1, x_2, \dots, x_k\}$ is orthogonal set then

$$\|x_1 + x_2 + \cdots + x_k\|^2 = \|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_k\|^2$$

Proof: by induction, the claim of the corollary is true for $k = 1, 2$ by Thm. Suppose true for some j and study $j+1$, since $(x_1 + \cdots + x_j)\cdot x_{j+1} = x_1 \cdot x_{j+1} + \cdots + x_j \cdot x_{j+1} = 0 + \cdots + 0 = 0.$

$$\begin{aligned} \|x_1 + \cdots + x_j + x_{j+1}\|^2 &= \|x_1 + \cdots + x_j\|^2 + \|x_{j+1}\|^2 \\ &= \|x_1\|^2 + \cdots + \|x_j\|^2 + \|x_{j+1}\|^2 \quad \text{by induction hypothesis.} \end{aligned}$$

Thus the Corollary follows by induction. //

Thm/ If $\beta = \{v_1, \dots, v_n\}$ is an orthonormal basis for $\mathbb{R}^n$ and $x = \sum_{i=1}^{n} c_i v_i$ then

$$\|x\|^2 = c_1^2 + c_2^2 + \cdots + c_n^2 = \|[x]_\beta\|^2$$

Proof: notice $v_i \cdot v_j = \delta_{ij} \Rightarrow (c_i v_i)\cdot(c_j v_j) = c_i c_j \delta_{ij}$ thus $\{c_1 v_1, c_2 v_2, \dots, c_n v_n\}$ is orthogonal and so,

$$\begin{aligned} \|x\|^2 &= \|c_1 v_1\|^2 + \|c_2 v_2\|^2 + \cdots + \|c_n v_n\|^2 \\ &= |c_1|^2 + |c_2|^2 + \cdots + |c_n|^2 \quad \text{since } \|c_j v_j\| = |c_j|\,\cancelto{1}{\|v_j\|} = |c_j| \\ &= c_1^2 + c_2^2 + \cdots + c_n^2 = \|[x]_\beta\|^2. // \end{aligned}$$

How to create orthogonal sets

There is a method to take a LI subset of column vectors and replace it with an orthogonal subset whose span gives the same subspace as the given subset.

$$\boxed{\begin{array}{c} W = \text{span}(S) \\ S \text{ LI} \end{array}} \xrightarrow{\text{G.S.A.}} \boxed{\begin{array}{c} W = \text{span}(S') \\ S' \text{ orthogonal} \end{array}}$$

The Gram Schmidt Algorithm (GSA) is this method. It's based on formulas stemming from the <u>vector</u> <u>projection</u>

Def^{n}/ Let $v, w \in \mathbb{R}^n$ then $\text{Proj}_v(w) = \left(\frac{w \cdot v}{v \cdot v}\right) v$

and $\text{Orth}_v(w) = w - \text{Proj}_v(w)$. ($v \neq 0$ assumed)

$\boxed{\text{E2}}$ $v = (1, 2, 3, 4)$, $w = (0, 1, 3, 0)$

$w \cdot v = 0 + 2 + 9 + 0 = 11$ and $v \cdot v = 1 + 4 + 9 + 16 = 30$

$$\text{Proj}_v(w) = \left(\frac{11}{30}\right)(1, 2, 3, 4) = \left(\frac{11}{30}, \frac{22}{30}, \frac{33}{30}, \frac{44}{30}\right)$$

$$\text{Orth}_v(w) = w - \text{Proj}_v(w) = \left(\frac{-11}{30}, \frac{8}{30}, \frac{57}{30}, \frac{-44}{30}\right)$$

You can check $\text{Proj}_v(w) \cdot \text{Orth}_v(w) = 0$ and $\text{Proj}_v(w) + \text{Orth}_v(w) = w$. Since $v \parallel \text{Proj}_v(w)$ we likewise see $v \perp \text{Orth}_v(w)$.

$$\{v_1 = (1,2,3,4),\ v_2 = (0,1,3,0)\} \longmapsto \{v_1' = v_1,\ v_2' = \text{Orth}_{v_1}(w)\}$$

$$\text{span}\underbrace{\{v_1, v_2\}}_{\text{LI}} = \text{span}\underbrace{\{v_1', v_2'\}}_{\text{orthogonal}}$$

<u>Remark:</u> if we want to create an orthonormal set we can go one more step and normalize the vectors in $S' = \{v_1', v_2'\}$ by dividing by $\|v_1'\|$ and $\|v_2'\|$,