LECTURE 1: SYSTEMS OF LINEAR EQ^{n}s AND ROW REDUCTION

Given a system of linear equations we can use row-reduction to find solutions. To do this we remove the variables from explicit mention and focus our calculations on a box of #'s called a <u>matrix</u>

$\boxed{E1}$ $\begin{pmatrix} x + y = 13 \\ x - y = -1 \end{pmatrix}$

$+: \quad 2x = 12 \Rightarrow x = 6$

$-: \quad 2y = 14 \Rightarrow y = 7 \quad \therefore \boxed{(6,7) \text{ is solution}}$

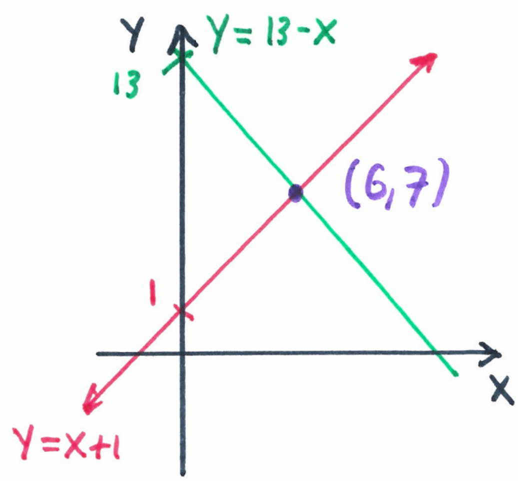
$$\left[\begin{array}{cc|c} 1 & 1 & 13 \\ 1 & -1 & -1 \end{array}\right] \xrightarrow{r_2 + r_1} \left[\begin{array}{cc|c} 1 & 1 & 13 \\ 2 & 0 & 12 \end{array}\right] \longrightarrow 2x = 12 \quad \therefore \underline{x = 6}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 13 \\ 1 & -1 & -1 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{cc|c} 1 & 1 & 13 \\ 0 & -2 & -14 \end{array}\right] \longrightarrow -2y = -14 \quad \therefore \underline{y = 7}$$

We could also use a graphical solution, but this is generally lacking precision

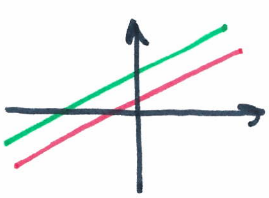
$x + y = 13 \Rightarrow \underline{y = 13 - x}$

$x - y = -1 \Rightarrow \underline{y = x + 1}$

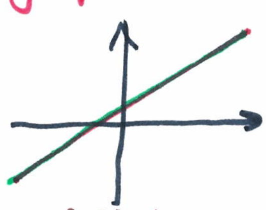


(unique solution)

Remark: we can already see all three possible types of solution sets with two-dim'l graphs:



inconsistent (no solution)



(infinitely many solutions)

consistent

E2

$$\begin{aligned} x + y + 2z &= 5 \\ 2y + 3z &= 9 \\ 2x - 4y + z &= 1 \end{aligned}$$

Solve via the row-reduction technique

$$\begin{array}{cccc} (x & y & z) & \\ \end{array}$$

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 2 & 3 & 9 \\ 2 & -4 & 1 & 1 \end{array}\right]}_{\text{augmented coefficient matrix}} \xrightarrow{r_3 - 2r_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 2 & 3 & 9 \\ 0 & -6 & -3 & -9 \end{array}\right]$$

$[A|b]$

$$\xrightarrow[3r_2]{6r_1} \left[\begin{array}{ccc|c} 6 & 6 & 12 & 30 \\ 0 & 6 & 9 & 27 \\ 0 & -6 & -3 & -9 \end{array}\right] \xrightarrow[r_3 + r_2]{r_1 - r_2} \left[\begin{array}{ccc|c} 6 & 0 & 3 & 3 \\ 0 & 6 & 9 & 27 \\ 0 & 0 & 6 & 18 \end{array}\right] *$$

$$\xrightarrow{r_3/6} \left[\begin{array}{ccc|c} 6 & 0 & 3 & 3 \\ 0 & 6 & 9 & 27 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[r_2 - 9r_3]{r_1 - 3r_3} \left[\begin{array}{ccc|c} 6 & 0 & 0 & -6 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[r_2/6]{r_1/6} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right]}_{\text{rref } [A|b]} \quad \begin{aligned} &\Rightarrow x = -1 \\ &\Rightarrow y = 0 \\ &\Rightarrow z = 3 \end{aligned}$$

(reduced-row-echelon-form)

$\therefore$ $\boxed{(-1, 0, 3) \text{ is the solution}}$

Remark: once the matrix is reduced to rref-format it becomes immediately clear what the solution is. Notice, we could have stopped at * and solved the system with a few easy calculations since the 3rd row showed $6z = 18 \rightarrow \underline{z = 3}$ then we could eliminate z from the other two eq^{n}'s and solve the easier 2 eq^{n}'s and 2 unknowns problem.

E3

$$
\begin{aligned}
x_2 + x_4 &= 10\\
x_1 - x_2 + 3x_4 &= 20\\
3x_1 - x_3 + x_4 &= 0
\end{aligned}
$$

Find the solution using row reduction

$$
[A|b] = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 10\\ 1 & -1 & 0 & 3 & 20\\ 3 & 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 3 & 20\\ 0 & 1 & 0 & 1 & 10\\ 3 & 0 & -1 & 1 & 0 \end{array}\right]
$$

$$
\xrightarrow{r_3 - 3r_1} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 3 & 20\\ 0 & 1 & 0 & 1 & 10\\ 0 & 3 & -1 & -8 & -60 \end{array}\right] \xrightarrow{r_3 - 3r_2} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 3 & 20\\ 0 & 1 & 0 & 1 & 10\\ 0 & 0 & -1 & -11 & -90 \end{array}\right]
$$

$$
\xrightarrow{-r_3} \underbrace{\left[\begin{array}{cccc|c} 1 & -1 & 0 & 3 & 20\\ 0 & 1 & 0 & 1 & 10\\ 0 & 0 & 1 & 11 & 90 \end{array}\right]}_{\text{ref}(A)} \xrightarrow{r_1 + r_2} \underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 30\\ 0 & 1 & 0 & 1 & 10\\ 0 & 0 & 1 & 11 & 90 \end{array}\right]}_{\text{rref}(A)} \begin{array}{c} ① \\ ② \\ ③ \end{array}
$$

ref(A): "reduced echelon form" (forward pass complete)

rref(A): clearing out entries above pivot positions constitutes the backwards pass, just one step here.

The solution here involves a free parameter, I'll use x_4 since it is the non-pivotal variable.

① : $x_1 + 4x_4 = 30 \quad \therefore \quad x_1 = 30 - 4x_4$

② : $x_2 + x_4 = 10 \quad \therefore \quad x_2 = 10 - x_4$

③ : $x_3 + 11x_4 = 90 \quad \therefore \quad x_3 = 90 - 11x_4$

$$
\boxed{\text{Solution Set} = \{(30 - 4x_4,\ 10 - x_4,\ 90 - 11x_4,\ x_4) \mid x_4 \in \mathbb{R}\}}
$$

E4

$$x_1 + x_2 = 3$$
$$x_2 + x_3 = 5$$
$$x_3 + x_4 = 7$$
$$x_1 - x_2 = -1$$

find the solution using row reduction

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 & 7 \\ 1 & -1 & 0 & 0 & -1 \end{array}\right] \xrightarrow{r_4 - r_1} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & -2 & 0 & 0 & -4 \end{array}\right]$$

$$\begin{array}{c} \xrightarrow{r_1 + r_4/2} \\ \xrightarrow{r_2 + r_4/2} \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & -2 & 0 & 0 & -4 \end{array}\right] \begin{array}{c} \xrightarrow{r_3 - r_2} \\ \xrightarrow{r_4/-2} \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad \therefore \boxed{\text{solution is } (1,2,3,4)}$$

this notation indicates several row operations, or perhaps just one, but the details are left to the reader. Use this at your own risk, it is the destroyer of partial credit.

$\boxed{E5}$ Find all quadratic polynomials whose graphs contain $(-2,3)$, $(-1,4)$, $(0,2)$, $(1,5)$.

Consider $f(x) = Ax^2 + Bx + C$, we wish to find possible choices for A, B, C for which

$$\underbrace{f(-2)=3}_{4A-2B+C=3},\quad \underbrace{f(-1)=4}_{A-B+C=4},\quad \underbrace{f(0)=2}_{C=2},\quad \underbrace{f(1)=5}_{A+B+C=5}$$

Let's be wise and substitute $C=2$ into the other equations to obtain,

$$\begin{aligned} 4A-2B&=0 \\ A-B&=1 \\ A+B&=2 \end{aligned} \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 4 & -2 & 0 \end{array}\right] \xrightarrow[r_3-4r_1]{r_2-r_1} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & -4 \end{array}\right]$$

$$\xrightarrow{r_3-r_2} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -5 \end{array}\right] \sim \underbrace{\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]}$$

game over.

$$\underbrace{0\cdot A + 0\cdot B = -5}_{\text{impossible eq}^{\underline{n}}}$$

there is a pivot column in the inhomogeneous column, this signals the system is <u>inconsistent</u>

$\therefore$ <u>NO SOLUTION EXISTS.</u>

($\nexists$ such a quadratic polynomial)