LECTURE 20: VECTOR PROJECTIONS, G.S.A., QR-DECOMPOSITION

We saw how Proj and Orth can be used to take a LI pair of vectors v, w and create an orthogonal pair $v, \text{Orth}_v(w) = w - \text{Proj}_v(w) = v_1', v_2'$ and we can verify $\text{span}\{v, w\} = \text{span}\{v_1', v_2'\}$. What if we have LI $\{v_1, v_2, v_3\} = S$ how to create $S' = \{v_1', v_2', v_3'\}$ orthogonal with $\text{span}\, S = \text{span}\, S'$?

1.) $v_1' = v_1$

2.) $v_2' = \text{Orth}_{v_1'}(v_2) = v_2 - \left(\frac{v_1' \cdot v_2}{v_1' \cdot v_1'}\right) v_1'$

3.) $v_3' = \text{Orth}_{v_1'}\left(\text{Orth}_{v_2'}(v_3)\right)$

$$= \text{Orth}_{v_1'}\left(v_3 - \left(\frac{v_2' \cdot v_3}{v_2' \cdot v_2'}\right) v_2'\right)$$

$$= v_3 - \left(\frac{v_2' \cdot v_3}{v_2' \cdot v_2'}\right) v_2' - \frac{1}{v_1' \cdot v_1'}\left[v_1' \cdot \left[v_3 - \left(\frac{v_2' \cdot v_3}{v_2' \cdot v_2'}\right) v_2'\right]\right] v_1'$$

$$= v_3 - \left(\frac{v_2' \cdot v_3}{v_2' \cdot v_2'}\right) v_2' - \left(\frac{v_1' \cdot v_3}{v_1' \cdot v_1'}\right) v_1' \qquad v_1' \cdot v_2' = 0$$

Continuing in this fashion, given $S = \{v_1, v_2, \ldots, v_n\}$ LI we can create orthogonal $S' = \{v_1', v_2', \ldots, v_n'\}$ with $\text{span}(S) = \text{span}(S')$ via the following algorithm

Def^n/ Gram-Schmidt-Algorithm

1.) Let $v_1' = v_1$

2.) Let $v_k' = v_k - \sum_{i=1}^{k-1} \left(\frac{v_i' \cdot v_k}{v_i' \cdot v_i'}\right) v_i'$ for $k = 2, \ldots, n$

$\boxed{E1}$ $V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \end{bmatrix}$, $V_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$,

Find orthogonal basis for $W = \text{span}\{V_1, V_2, V_3\}$

$$V_1' = (1, 1, 1, 0)$$

$$V_2' = V_2 - \left(\frac{V_1' \cdot V_2}{V_1' \cdot V_1'}\right) V_1' = (3, 0, 4, 0) - \frac{7}{3}(1, 1, 1, 0) = \underbrace{(2/3, -7/3, 5/3, 0)}_{V_2'}$$

Next, note $V_1' \cdot V_3 = 0$ and $V_2' \cdot V_3 = \frac{-12}{3}$

and since $V_2' \cdot V_2' = \frac{4 + 49 + 25}{9} = \frac{78}{9}$ we obtain

$$\frac{V_2' \cdot V_3}{V_2' \cdot V_2'} = \frac{-12/3}{78/9} = \frac{-6}{13} \text{ thus}$$

$$V_3' = (0, 1, -1, 0) + \frac{6}{13} V_2' = (0, 1, -1, 0) + \frac{2}{13}(2, -7, 5, 0)$$

$$\therefore V_3' = (4/13, -1/13, -3/13, 0)$$

$$\{V_1', V_2', V_3'\} = \underbrace{\left\{(1, 1, 1, 0), \tfrac{1}{3}(2, -7, 5, 0), \tfrac{1}{13}(4, -1, -3, 0)\right\}}_{\text{orthogonal.}}$$

<u>Remark</u>: I suspect we'll have much better success if we make a habit of modifying the GSA to suppress fractions along the way. For example, in $\boxed{E1}$ we could set $V_2' = (2, -7, 5, 0)$ then

$$V_3' = (0, 1, -1, 0) - \left(\frac{(2, -7, 5, 0) \cdot (0, 1, -1, 0)}{78}\right)(2, -7, 5, 0)$$

$$= (0, 1, -1, 0) - \frac{2}{13}(2, -7, 5, 0)$$

Ultimately we can rescale each vector and maintain orthogonality since $V \cdot W = 0 \iff (c_1 V) \cdot (c_2 W) = 0$.
for $c_1, c_2 \neq 0$

$\boxed{E2}$ once more consider $V_1 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix}$, $V_2 = \begin{bmatrix}3\\0\\4\\0\end{bmatrix}$, $V_3 = \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$ and now find an <u>orthonormal</u> basis for $W = \text{span}\{V_1, V_2, V_3\}$

From $\boxed{E1}$ we can rescale our G.S.A. S' to give

$$S' = \{\underbrace{(1,1,1,0)}_{\text{length } \sqrt{3}}, \underbrace{(2,-7,5,0)}_{\text{length } \sqrt{78}}, \underbrace{(4,-1,-3,0)}_{\text{length } \sqrt{26}}\}$$

$$\therefore \boxed{S'' = \left\{\tfrac{1}{\sqrt{3}}(1,1,1,0), \tfrac{1}{\sqrt{78}}(2,-7,5,0), \tfrac{1}{\sqrt{26}}(4,-1,-3,0)\right\}}$$

(orthonormal basis for W

Remark: we can extend S'' to an orthonormal basis for $\mathbb{R}^4$, can you see how w/o any additional calculation?

Defn/ Given a set of nonzero vectors $\{V_1, V_2, .., V_n\}$ we <u>normalize</u> the set by rescaling each vector to be a unit-vector $\left\{\frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|}, \dots, \frac{V_n}{\|V_n\|}\right.$.

If we normalize an orthogonal set then we create an orthonormal set (provided we can normalize, can't have the zero vector in the set which is given)

$\boxed{E3}$ $\{(1,1,0,0), (1,-1,1,1), (0,0,0,0)\} = S$

this is an orthogonal set, however we cannot normalize the whole set. If $S_2 = \{(1,1,0,0), (1,-1,1,1)\}$ then $S_3 = \left\{\frac{1}{\sqrt{2}}(1,1,0,0), \frac{1}{2}(1,-1,1,1)\right\}$ is <u>orthonormal</u>

<u>QUESTION</u>: can you see how to extend S_3 to an orthonormal basis for $\mathbb{R}^4$? What to put

$$\left\{\tfrac{1}{\sqrt{2}}(1,1,0,0), \tfrac{1}{2}(1,-1,1,1), \_\_\_\_, \_\_\_\_\right\}$$

E4 $V_1 = \begin{bmatrix}1\\0\\1\\0\\1\end{bmatrix}$ $V_2 = \begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix}$ $V_3 = \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$ Calculate orthonormal basis for $W = \text{span}\{V_1, V_2, V_3\}$

Since $V_1 \cdot V_2 = 0$ we can jump to orthogonalizing the third vector, (think $V_1' = V_1$ and $V_2' = V_2$)

$$V_3' = V_3 - \left(\frac{V_1 \cdot V_3}{V_1 \cdot V_1}\right) V_1 - \left(\frac{V_2 \cdot V_3}{V_2 \cdot V_2}\right) V_2$$

$$= (1,2,3,4,5) - \frac{9}{3} V_1 - \frac{6}{2} V_2$$

$$= (1,2,3,4,5) - 3(1,0,1,0,1) - 3(0,1,0,1,0)$$

$$= (1,2,3,4,5) - (3,3,3,3,3)$$

$$= (-2,-1,0,1,2)$$

Normalizing we find, orthonormal basis for W of:

$$\boxed{\left\{ \frac{1}{\sqrt{3}}(1,0,1,0,1),\ \frac{1}{\sqrt{2}}(0,1,0,1,0),\ \frac{1}{\sqrt{10}}(-2,-1,0,1,2) \right\}}$$

<u>Question</u>: if the vectors above are u_1, u_2, u_3 then can you find u_4, u_5 s.t. $\{u_1, u_2, u_3, u_4, u_5\}$ is an orthonormal basis for $\mathbb{R}^5$? (maybe we should find a method to answer such questions...)

E5 Let $V_1 = \begin{bmatrix}1\\2\end{bmatrix}$ and $V_2 = \begin{bmatrix}3\\4\end{bmatrix}$ then if we run the GSA on these we'll find something we could easily have guessed.

$$V_2' = V_2 - \left(\frac{V_1 \cdot V_2}{V_1 \cdot V_1}\right) V_1 = \begin{bmatrix}3\\4\end{bmatrix} - \frac{11}{5}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}4/5\\-2/5\end{bmatrix} = \frac{2}{5}\begin{bmatrix}2\\-1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2\\-1\end{bmatrix} \perp \begin{bmatrix}1\\2\end{bmatrix} \quad \text{yep.}$$

$\boxed{E6}$ Let's be lazy when we can,
to orthonormalize $v_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 6 \\ 7 \\ 0 \end{bmatrix}$,

simply use $v_1' = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$ and $v_2' = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$

thus $\left\{ \frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \frac{1}{5}\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\} \leftarrow$ orthonormal

QR- DECOMPOSITION OF MATRIX

Th^m/ Given a matrix A with LI columns we can find Q and R for which $A = QR$ and $Q^T Q = I$ while R is an upper triangular matrix with positive entries on its diagonal.

In particular, for $A \in \mathbb{R}^{m \times n}$ we have $Q \in \mathbb{R}^{m \times n}$ whereas $R \in \mathbb{R}^{n \times n}$ and the columns of Q form an <u>orthonormal</u> basis for $\mathrm{Col}(A)$.

$\boxed{E7}$ $A = \begin{bmatrix} 3 & 6 \\ 4 & 7 \\ 0 & 0 \end{bmatrix} = QR \Rightarrow Q^T A = Q^T Q R = I_2 R = R$

from $\boxed{E6}$, $Q^T A = \frac{1}{5}\begin{bmatrix} 3 & 4 & 0 \\ 4 & -3 & 0 \end{bmatrix}\begin{bmatrix} 3 & 6 \\ 4 & 7 \\ 0 & 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 25 & 46 \\ 0 & 3 \end{bmatrix}$

$$A = \underbrace{\begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \\ 0 & 0 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 5 & 46/5 \\ 0 & 3/5 \end{bmatrix}}_{R} \leftarrow \text{QR DECOMPOSITION OF } A$$

$\boxed{E8}$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 1 & 4 \\ 1 & 0 & 5 \end{bmatrix} \quad \text{found} \quad Q = \left[\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \middle| \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \middle| \frac{1}{\sqrt{10}}\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right]$$

Let $\alpha = 1/\sqrt{3}$, $\beta = 1/\sqrt{2}$, $\gamma = 1/\sqrt{10}$ then for $A = QR$ we have $Q^T A = Q^T Q R = IR = R$ thus,

$$R = Q^T A = \begin{bmatrix} \alpha & 0 & \alpha & 0 & \alpha \\ 0 & \beta & 0 & \beta & 0 \\ -2\gamma & -\gamma & 0 & \gamma & 2\gamma \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 1 & 4 \\ 1 & 0 & 5 \end{bmatrix}$$

$$R = \begin{bmatrix} 3\alpha & 0 & 9\alpha \\ 0 & 2\beta & 6\beta \\ 0 & 0 & 10\gamma \end{bmatrix}$$

$$A = \begin{bmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{10} \\ 0 & 1/\sqrt{2} & -1/\sqrt{10} \\ 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{10} \\ 1/\sqrt{3} & 0 & 2/\sqrt{10} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 3\sqrt{3} \\ 0 & \sqrt{2} & 3\sqrt{2} \\ 0 & 0 & \sqrt{10} \end{bmatrix}$$

THE ORTHOGONAL BASIS EXTENSION PROBLEM

We'd like to find a method to take an orthogonal set like $\{v_1, v_2, \dots, v_k\} \subseteq \mathbb{R}^n$ and extend it to a set $\{v_1, v_2, \dots, v_k, \underbrace{v_{k+1}, \dots, v_n}\}$ which is likewise orthogonal

How to find these?

Def^n/ Given $S \subseteq \mathbb{R}^n$ the "perp" or orthogonal complement of S is denoted $S^\perp$ and it is defined by

$$S^\perp = \{x \in \mathbb{R}^n \mid x \cdot s = 0 \ \forall s \in S\}$$

Our problem is to calculate an orthogonal basis for $\{v_{k+1}, \dots, v_n\}^\perp$. We should be more general for our future convenience.

Th^m/ If $S = \{s_1, s_2, \dots, s_k\} \subseteq \mathbb{R}^n$ then $S^\perp = \text{Null}([S]^T)$

Proof: suppose $x \in S^\perp$ then $s_j \cdot x = 0 \ \forall j = 1, 2, \dots, k$ thus $s_j^T x = 0$ for $j = 1, 2, \dots, k$. Observe,

$$\begin{aligned}
\text{Null}([S]^T) &= \{x \in \mathbb{R}^n \mid [s_1|s_2|\cdots|s_k]^T x = 0\} \\
&= \left\{x \in \mathbb{R}^n \;\middle|\; \begin{bmatrix} s_1^T x \\ s_2^T x \\ \vdots \\ s_k^T x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\} \\
&= \{x \in \mathbb{R}^n \mid s_j^T x = 0 \ \forall j = 1, 2, \dots, k\} \\
&= S^\perp.
\end{aligned}$$

Remark: we already know how to find the basis for Null(A) hence Null $[S]^T$ is something we can calculate the basis for... then we'd just need to run G.S.A. to orthogonalize it.

$\boxed{E9}$ We had $\tilde{S}_3 = \{(1,1,0,0), (1,-1,1,1)\}$ (I removed $1/\sqrt{2}$ and $1/2$ for now) let's find basis for $\tilde{S}_3^{\perp}$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & -1/2 \end{bmatrix}$$

$$x \in \text{Null}\,[\tilde{S}_3]^T \Rightarrow \begin{array}{l} x_1 = -(x_3 + x_4)\frac{1}{2} \\ x_2 = (x_3 + x_4)\frac{1}{2} \end{array}$$

$$x = \left(\frac{-x_3}{2} - \frac{x_4}{2}, \frac{x_3}{2} + \frac{x_4}{2}, x_3, x_4\right)$$

$$= (x_3/2)(-1, 1, 2, 0) + (x_4/2)(-1, 1, 0, 2)$$

Thus $\beta = \{(-1,1,2,0), (-1,1,0,2)\}$ is basis of $\tilde{S}_3^{\perp}$

But β is not orthogonal, we can fix this,

$$u_1 = (-1,1,2,0), \quad u_2 = (-1,1,0,2)$$

$$u_2' = u_2 - \left(\frac{u_1 \cdot u_2}{u_1 \cdot u_1}\right) u_1 = (-1,1,0,2) - \frac{2}{6}(-1,1,2,0)$$

$$u_2' = (-1 + 1/3, 1 - 1/3, -2/3, 2) = (-2/3, 2/3, -2/3, 6/3)$$

Hence, $\{(-1,1,2,0), (-1,1,-1,3)\}$ is the desired orthogonal basis for $\tilde{S}_3^{\perp}$ and to answer the question posed after E3,

$$\left\{\frac{1}{\sqrt{2}}(1,1,0,0), \frac{1}{2}(1,-1,1,1), \frac{1}{\sqrt{6}}(-1,1,2,0), \frac{1}{\sqrt{12}}(-1,1,-1,3)\right\}$$

NEXT: how to apply orthonormal bases to solve problems of geometry and algebra.