Lecture 23: Calculus of Matrices & Diagonalization

In this Lecture I'll introduce the calculus of matrix-valued functions of time, then discuss systems of ODEs briefly and explain how diagonalization helps solve such systems.

Def If $A : \mathbb{R} \to \mathbb{R}^{m \times n}$ has <u>component functions</u> $A_{ij} : \mathbb{R} \to \mathbb{R}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ then we define $\frac{dA}{dt} : \mathbb{R} \to \mathbb{R}^{m \times n}$ to be the function from $\mathbb{R} \to \mathbb{R}^{m \times n}$ with component functions $\frac{dA_{ij}}{dt} : \mathbb{R} \to \mathbb{R}$. That is,

$$\left(\frac{dA}{dt}\right)_{ij} = \frac{dA_{ij}}{dt}$$

In other words, we differentiate a matrix component-wise. We can parse the derivative many ways,

$$\frac{dA}{dt} = \left[\frac{d}{dt}[col_1 A] \Big| \cdots \Big| \frac{d}{dt}[col_n(A)] \right] = \left[\begin{array}{c} \frac{d}{dt}[row_1(A)] \\ \hline \vdots \\ \hline \frac{d}{dt}[row_m(A)] \end{array} \right]$$

Calculus of matrices has to face noncommutative structure of matrix multiplication.

E1 $A = \begin{bmatrix} 1 & t \\ t^2 & 2 \end{bmatrix}$ $\quad A^2 = \begin{bmatrix} 1 & t \\ t^2 & 2 \end{bmatrix}\begin{bmatrix} 1 & t \\ t^2 & 2 \end{bmatrix} = \left[\begin{array}{c|c} 1+t^3 & 3t \\ t^2+2t^2 & 4+t^3 \end{array}\right]$

$$\frac{dA}{dt} = \begin{bmatrix} 0 & 1 \\ 2t & 0 \end{bmatrix} \qquad \frac{d}{dt}(A^2) = \left[\begin{array}{c|c} 3t^2 & 3 \\ 6t & 3t^2 \end{array}\right]$$

$$A\frac{dA}{dt} = \begin{bmatrix} 1 & t \\ t^2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2t & 0 \end{bmatrix} = \begin{bmatrix} 2t^2 & 1 \\ 4t & t^2 \end{bmatrix}$$

$$\frac{dA}{dt}A = \begin{bmatrix} 0 & 1 \\ 2t & 0 \end{bmatrix}\begin{bmatrix} 1 & t \\ t^2 & 2 \end{bmatrix} = \begin{bmatrix} t^2 & 2 \\ 2t & 2t^2 \end{bmatrix}$$

Thus we see $\frac{d}{dt}(A^2) \neq 2A\frac{dA}{dt}$, rather $\underline{\frac{d}{dt}(A^2) = A\frac{dA}{dt} + \frac{dA}{dt}A}$.

PROPOSITION:

Let A, B be matrices of functions of t and suppose c is a constant and v is constant,

(1.) $\dfrac{d}{dt}(A+B) = \dfrac{dA}{dt} + \dfrac{dB}{dt}$

(2.) $\dfrac{d}{dt}(AB) = \dfrac{dA}{dt}B + A\dfrac{dB}{dt}$

(3.) $\dfrac{d}{dt}(cA) = c\dfrac{dA}{dt}$ and $\dfrac{d}{dt}(Av) = \dfrac{dA}{dt}v$

The proof of the above assertions follow from Calculus I,

(1.)
$$\left(\frac{d}{dt}(A+B)\right)_{ij} = \frac{d}{dt}(A+B)_{ij} = \frac{d}{dt}\left(A_{ij}+B_{ij}\right) = \frac{dA_{ij}}{dt} + \frac{dB_{ij}}{dt} = \left(\frac{dA}{dt} + \frac{dB}{dt}\right)_{ij}.$$

(2.)
$$\begin{aligned}\left(\frac{d}{dt}(AB)\right)_{ij} &= \frac{d}{dt}(AB)_{ij} \\ &= \frac{d}{dt}\left(\sum_k A_{ik}B_{kj}\right) \\ &= \sum_k \frac{d}{dt}\left(A_{ik}B_{kj}\right) \\ &= \sum_k \left(\frac{dA}{dt}\right)_{ik} B_{kj} + \sum_k A_{ik}\left(\frac{dB}{dt}\right)_{kj} \\ &= \left(\frac{dA}{dt}B + A\frac{dB}{dt}\right)_{ij}\end{aligned}$$

Then (3.) follows from particular cases of (2.).

Def^n/ Given an $(n \times n)$ constant matrix A we say $\dfrac{dx}{dt} = Ax$ is a <u>system</u> of n <u>first order</u> Ordinary Differential Equations (ODE) with coefficient matrix A. A <u>solution</u> x is a function from $\mathbb{R}$ to $\mathbb{R}^n$ for which $\dfrac{dx}{dt} = Ax$.

$\boxed{E2}$ $\left\{ \begin{aligned} \frac{dx_1}{dt} &= x_1 - x_2 \\ \frac{dx_2}{dt} &= -x_1 + x_2 \end{aligned} \right\} \longrightarrow \frac{dx}{dt} = Ax, \quad A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Here there is an easy trick to solve the system, try adding the differential eq^{n}s,

$$\frac{dx_1}{dt} + \frac{dx_2}{dt} = (x_1 - x_2) + (-x_1 + x_2) = 0$$

$$\therefore \quad \frac{d}{dt}(x_1 + x_2) = 0 \quad \Longrightarrow \quad \underline{x_1 + x_2 = c_1}.$$

$$\therefore x_2 = c_1 - x_1$$

$$\frac{dx_1}{dt} - \frac{dx_2}{dt} = (x_1 - x_2) - (x_2 - x_1) = 2(x_1 - x_2)$$

$$\frac{d}{dt}(x_1 - x_2) = 2(x_1 - x_2)$$

$$x_1 - x_2 = c_2 e^{2t}$$

$$x_1 + x_2 = c_1$$

$$(+) \quad \underline{x_1 = \tfrac{1}{2}(c_1 + c_2 e^{2t})}.$$

$$(-) \quad x_2 = \tfrac{1}{2}(c_1 - c_2 e^{2t})$$

$$\underline{x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{c_1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}.$$

<u>Remark</u>: $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

In Calculus II we saw $\frac{dy}{dt} = \lambda y$ has solution given by separation of variables

$$\int \frac{dY}{Y} = \int \lambda\, dt \rightarrow \ln|Y| = \lambda t + C \quad \therefore \quad |Y| = e^c e^{\lambda t}$$

$$\Rightarrow \underline{y = c_1 e^{\lambda t}}.$$

Now we face a similar problem to solve $\frac{dx}{dt} = Ax$ where $A \in \mathbb{R}^{n \times n}$.

PROPOSITION:

$x = e^{\lambda t} u$ solves $\frac{dx}{dt} = Ax \Rightarrow (A - \lambda I)u = 0$.

Proof: Suppose $x = e^{\lambda t} u$ solves $\frac{dx}{dt} = Ax$ then

$$\frac{d}{dt}(e^{\lambda t} u) = \lambda e^{\lambda t} u + e^{\lambda t} \underbrace{\frac{du}{dt}}_{0} = A(e^{\lambda t} u)$$

$$\therefore \quad \lambda u = Au \Rightarrow \underline{(A - \lambda I)u = 0}. //$$

Def^n/ If $A \in \mathbb{R}^{n \times n}$ and there is a constant λ for which $(A - \lambda I)u = 0$ for some <u>nonzero</u> vector u then λ is an <u>eigenvalue</u> and u is an <u>eigenvector</u> with e-value λ for A.

PROPOSITION:

$\frac{dx}{dt} = Ax$ has solution $x = e^{\lambda t} u$ with $u \neq 0$ if and only if $\det(A - \lambda I) = 0$. In other words, eigenvalues are necessarily zeros of the <u>characteristic equation</u> $\det(A - \lambda I) = 0$

Proof: $\det(A - \lambda I) = 0 \Leftrightarrow \mathrm{Null}(A - \lambda I) \neq \{0\}$

$\Leftrightarrow$ there exists $u \neq 0$ s.t. $(A - \lambda I)u = 0$.

Def^n/ If $A \in \mathbb{R}^{n\times n}$ and $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis of eigenvectors of A then β is an <u>eigenbasis</u> for A

Something very neat happens when we have an eigenbasis for the standard matrix of $T : \mathbb{R}^n \to \mathbb{R}^n$.
Suppose $[T] = A$ and suppose

$$T(v_1) = \lambda v_1, \quad T(v_2) = \lambda_2 v_2, \dots, T(v_n) = \lambda v_n$$

Then, if $\beta = \{v_1, v_2, \dots, v_n\}$ is basis,

$$[T]_{\beta,\beta} = \big[\, [\lambda_1 v_1]_\beta \,|\, [\lambda_2 v_2]_\beta \,|\, \cdots \,|\, [\lambda_n v_n]_\beta \,\big]$$
$$= [\lambda_1 e_1 \,|\, \lambda_2 e_2 \,|\, \cdots \,|\, \lambda_n e_n] = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

The matrix of T is <u>diagonalized</u>

<u>Remark</u>: $(A - \lambda I)v = 0 \iff Av = \lambda v \iff T(v) = \lambda v$ for $[T] = A$
(different ways to look at e-vectors)

Recall that

$$[T]_{\beta,\beta} = [\beta]^{-1}[T][\beta]$$

Th^m/ Let $A \in \mathbb{R}^{n\times n}$ then there exists invertible P for which $P^{-1}AP = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ iff $P = [v_1 | v_2 | \cdots | v_n]$ where $\det(P) \neq 0$ and $Av_j = \lambda_j v_j$ for $j = 1, 2, \dots, n$. That is, A is diagonalizable iff A has an eigenbasis β for which $[\beta]^{-1} A [\beta] = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ where $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of A.

How Diagonalization Makes Solution Easy

Suppose $A \in \mathbb{R}^{n\times n}$ and $Av_1 = \lambda_1 v_1, \dots, Av_n = \lambda_n v_n$ where $\beta = \{v_1, v_2, \dots, v_n\}$ is eigenbasis for A. Let $y = [\beta]^{-1}x = \Phi_\beta(x)$ define eigencoordinates for the system of ODEs $\frac{dx}{dt} = Ax$

Observe,

$$y = [\beta]^{-1}x \quad \text{then} \quad x = [\beta]y$$

$$\therefore \quad \frac{d}{dt}\left([\beta]y\right) = A[\beta]y$$

$$\Rightarrow [\beta]\frac{dy}{dt} = A[\beta]y$$

$$\Rightarrow \frac{dy}{dt} = [\beta]^{-1}A[\beta]y$$

$$\Rightarrow \frac{dy}{dt} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)y$$

$$\Rightarrow \frac{dy_i}{dt} = \lambda_i y_i \quad \therefore \quad \underline{y_i = c_i e^{\lambda_i t}}.$$

$$\Rightarrow x = [\beta]y = [v_1|v_2|\cdots|v_n]\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\Rightarrow x = y_1 v_1 + y_2 v_2 + \cdots + y_n v_n$$

$$\Rightarrow \boxed{x = c_1 e^{\lambda_1 t}v_1 + c_2 e^{\lambda_2 t}v_2 + \cdots + c_n e^{\lambda_n t}v_n}$$

Remark: IF we can find an eigenbasis for A then the eq$^{\underline{n}}$ above gives general sol$^{\underline{n}}$ for $\frac{dx}{dt} = Ax$.
(IF...)