Lecture 24 : Examples & Theory of Real Eigenvectors

We saw last time that solving $\frac{dx}{dt} = Ax$ is naturally accomplished via the use of eigenvectors of A. This Lecture focuses on the structure and calculation of such vectors.

Def$^{\underline{n}}$/ Given $A \in \mathbb{R}^{n \times n}$ if λ is an eigenvalue then we define $\mathcal{E}_\lambda = \text{Null}(A - \lambda I)$ to be the λ-eigenspace of A.

Our usual goal is to find bases for each eigenspace of a given matrix. The procedure goes as follows:

(1.) Calculate all roots of characteristic eq$^{\underline{n}}$ $\det(A - \lambda I)$

(2.) For each sol$^{\underline{n}}$ λ_j in (1.) find basis for $\text{Null}(A - \lambda_j I)$.

$\boxed{E1}$ $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ then $\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}$

$$= (\lambda - 1)^2 - 1$$
$$= (\lambda - 1 + 1)(\lambda - 1 - 1)$$
$$= \lambda(\lambda - 2) = 0$$
$$\underline{\lambda_1 = 0} \ \ \& \ \ \underline{\lambda_2 = 2}.$$

Consider then,

$$A - 0 \cdot I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \underline{\text{Null}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}}.$$

$$A - 2I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \underline{\text{Null}(A - 2I) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}}.$$

We find eigenbasis for A of $\beta = \left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

Remark: $[\beta]^{-1} A [\beta] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ (diagonalized)

Remark: whenever A^{-1} d.n.e. we know that $\text{Null}(A) \neq \{0\}$ hence $\lambda = 0$ is an eigenvalue. This makes factoring certain characteristic eq^{n}'s much easier.

$\boxed{E2}$ $A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -2 & -2 \\ 3 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x \in \text{Null}(A)$ has $x_1 = x_2 + x_3$

Thus $x \in \text{Null}(A)$ has $x = (x_2 + x_3, x_2, x_3)$ and so $\text{Null}(A) = \text{span}\{(1,1,0), (1,0,1)\} = \mathcal{E}_{\lambda=0}$

Theory will inform us soon that λ^2 must appear as factor in $\det(A - \lambda I)$. Let's work it out,

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -1 & -1 \\ 2 & -2-\lambda & -2 \\ 3 & -3 & -3-\lambda \end{bmatrix}$$

$$= (1-\lambda)\big[(\lambda+2)(\lambda+3) - 6\big] + \big[2(-3-\lambda) + 6\big] - 1\big[-6 + 3(\lambda+2)\big]$$

$$= (1-\lambda)\left[\lambda^2 + 5\lambda\right] + (-2\lambda) + 6 - 3\lambda + (-6)$$

$$= (1-\lambda)\lambda(\lambda+5) - 5\lambda$$

$$= \lambda\big((1-\lambda)(\lambda+5) - 5\big)$$

$$= \lambda(-\lambda^2 + 4\lambda).$$

$$= -\lambda^2(\lambda - 4)$$

It remains to calculate basis for $\mathcal{E}_{\lambda=4} = \text{Null}(A - 4I)$

$$A - 4I = \begin{bmatrix} -3 & -1 & -1 \\ 2 & -6 & -2 \\ 3 & -3 & -7 \end{bmatrix} \sim \begin{bmatrix} -6 & -2 & -2 \\ 6 & -18 & -6 \\ 6 & -6 & -14 \end{bmatrix} \sim \begin{bmatrix} -6 & -2 & -2 \\ 0 & -20 & -8 \\ 0 & -8 & -16 \end{bmatrix}$$

$$\sim \begin{bmatrix} -6 & -2 & -2 \\ 0 & -5 & -2 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} -6 & 0 & 2 \\ 0 & 0 & 8 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ what means ???

$\boxed{E2}$ continued, for $A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -2 & -2 \\ 3 & -3 & -3 \end{bmatrix}$ we claimed $\lambda_1 = 0$ or $\lambda_2 = 4$, yet $\text{Null}(A - 4I) = 0$ which goes to show $\det(A - 4I) \neq 0$!

$$\det(A - \lambda I) = -\lambda^2(\lambda + 4) = 0$$

the (−) last page was incorrect.

Then, $\lambda_2 = -4$ actually,

$$A + 4I = \begin{bmatrix} 5 & -1 & -1 \\ 2 & 2 & -2 \\ 3 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 5 & -1 & -1 \\ 5 & -1 & -1 \\ 3 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 5 & -1 & -1 \\ 3 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 15 & -3 & -3 \\ 15 & -15 & 5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 15 & -3 & -3 \\ 0 & -12 & 8 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 15 & -3 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 15 & 0 & -5 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{bmatrix} = \text{rref}(A + 4I)$$

Then for $x \in \text{Null}(A + 4I)$ we have $x_1 = \frac{x_3}{3}$ and $x_2 = \frac{2x_3}{3}$ so choose $x_3 = 3$ to obtain

$$\mathcal{E}_{\lambda_2 = -4} = \text{span}\{(1, 2, 3)\}$$

In summary, we find A has eigenbasis,

$$\beta = \{\underbrace{(1,1,0), (1,0,1)}_{\lambda_1 = 0}, \underbrace{(1,2,3)}_{\lambda_2 = -4}\}$$

<u>Remark</u>: $\text{trace}(A) = 1 - 2 - 3 = -4$ and $\det(A) = 0$. What do these have to do with λ_1 & λ_2?

$\boxed{E3}$ $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{pmatrix}$$
$$= (\lambda - 1)(\lambda + 1) - 3$$
$$= \lambda^2 - 4$$
$$= (\lambda + 2)(\lambda - 2) = 0 \Rightarrow \underline{\lambda_1 = -2,\ \lambda_2 = 2}.$$

$\underline{\lambda_1 = -2|}$ $A + 2I = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \underline{\mathcal{E}_1 = \text{span}\{(1,-3)\}}.$

$\underline{\lambda_2 = 2|}$ $A - 2I = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \underline{\mathcal{E}_2 = \text{span}\{(1,1)\}}.$

$\boxed{E4}$ $M = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & -1 \end{array}\right] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ (combining $\boxed{E1}$ and $\boxed{E3}$)

$$\det(M - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} \oplus \begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix}\right)$$
$$= \det\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} \det\begin{pmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{pmatrix}$$
$$= \lambda(\lambda - 2)(\lambda + 2)(\lambda - 2)$$
$$= \lambda(\lambda - 2)^2(\lambda + 2)$$

Use $\boxed{E1}$, $\boxed{E3}$

$$V_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad V_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad V_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad V_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix}$$

$\mathcal{E}_{\lambda=0} = \text{span}\{V_1\}$, $\mathcal{E}_{\lambda=2} = \text{span}\{V_2, V_3\}$, $\mathcal{E}_{\lambda=-2} = \text{span}\{V_4\}$

Up to now we've only seen diagonalizable examples. Each matrix in E1, E2, E3, E4 had an eigenbasis. Let's look at a few examples where $\nexists$ an eigenbasis over $\mathbb{R}$.

$\boxed{\text{E5}}$ Suppose $\theta \neq 0, \pi$ and $\theta \in (0, 2\pi)$ then

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ is a } \underline{\text{rotation}}\ \underline{\text{matrix}}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix}$$

$$= (\lambda - \cos\theta)^2 + \sin^2\theta$$

$$= 0 \implies \underline{\lambda = \cos\theta \pm i\sin\theta \notin \mathbb{R}}$$

Remark: in other words $\lambda = e^{\pm i\theta}$, it turns out A in $\boxed{\text{E5}}$ is diagonalizable over $\mathbb{C}$. Of course, $Av \neq \lambda v$ when Av is v rotated.

$\boxed{\text{E6}}$ $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = (\lambda - 3)^2 = 0$$

$$A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \implies \underline{\mathrm{Null}(A - 3I) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}}.$$

$\boxed{\text{E7}}$ $A = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 7-\lambda & 1 & 0 \\ 0 & 7-\lambda & 1 \\ 0 & 0 & 7-\lambda \end{pmatrix} = (7 - \lambda)^3 = 0$$

$$\mathrm{Null}(A - 7I) = \mathrm{Null}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \underline{\mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}}.$$

THEORY FOR EIGENVECTORS, EIGENVALUES AND E-SPACES

I'll begin with some results I don't intend to prove here but they're helpful guides to calculation.

Defn/ Suppose $A \in \mathbb{R}^{n \times n}$ has distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ such that $n_1 + n_2 + \dots + n_k = n$ and

$$\det(A - \lambda I) = (-1)^n (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_k)^{n_k}$$

then we say n_j is the <u>algebraic multiplicity</u> of λ_j for A for $j = 1, 2, \dots, k$. In contrast, $\dim(\mathrm{Null}(A - \lambda_j I)) = \dim(\mathcal{E}_{\lambda_j}) =$ <u>geometric multiplicity</u> of λ_j (let's say $\nu_j = \dim(\mathcal{E}_{\lambda_j})$)

In E1, E2, E3, E4 we had algebraic mult. = geom. mult. for each e-value. However, in E6 and E7 $n_j > \nu_j$

Thm/ Given A with all real eigenvalues, if eigenvalue λ_j has geometric multiplicity ν_j and algebraic multiplicity n_j then

$$1 \leq \nu_j \leq n_j$$

In other words, we can only find as many LI e-vectors as the algebraic mult. of the e-value. However, there is always at <u>least</u> one e-vector for each e-value.

I'll save some theory for future classes. That said, I think this $Th^{\underline{m}}$ and its proof are worth studying today.

$Th^{\underline{m}}$/ If A is diagonalizable and has eigenbasis $\beta = \{v_1, v_2, \dots, v_n\}$ with e-values $\lambda_1, \lambda_2, \dots, \lambda_n$ (possibly repeated) then

$$\text{trace}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$$

<u>PROOF</u>: Suppose $Av_j = \lambda_j v_j$ for $j = 1, 2, \dots, n$ where $\beta = \{v_1, v_2, \dots, v_n\}$ is LI and $A \in \mathbb{R}^{n \times n}$ and $\lambda_j \in \mathbb{R}$ for $j = 1, 2, \dots, n$. Then,

$$[\beta]^{-1} A [\beta] = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Therefore, as $\det([\beta]^{-1} A [\beta]) = \det([\beta][\beta]^{-1} A) = \det(A)$,

$$\det(A) = \det([\beta]^{-1} A [\beta]) = \det\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} = \underline{\lambda_1 \lambda_2 \cdots \lambda_n}.$$

and, likewise

$$\text{tr}(A) = \text{tr}([\beta]^{-1} A [\beta]) = \text{tr}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \underline{\lambda_1 + \cdots + \lambda_n} \quad //$$

<u>Remark</u>: this $Th^{\underline{m}}$ is nice to check work. Also, it actually still holds when the matrix is not diagonalizable or when the eigenvalues are complex. We'll see why later.