LECTURE 25: REAL EIGENVECTORS PART II

Let's review diagonalization, we should prove the following result:

> **Th^m/** Let $A \in \mathbb{R}^{n \times n}$. Then there exists an eigenbasis β for $A \iff$ there exists invertible $P \in \mathbb{R}^{n \times n}$ for which $P^{-1}AP = D$ where D is diagonal. Moreover, when A is diagonalizable we have $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A, possibly repeated.

Proof: Let $A \in \mathbb{R}^{n \times n}$. To prove $\iff$ we prove $\Rightarrow$ and $\Leftarrow$. Consider,

$\Rightarrow$ Suppose $\beta = \{V_1, V_2, \dots, V_n\}$ is eigenbasis where $AV_j = \lambda_j V_j$ for $j = 1, 2, \dots, n$. Construct $P = [\beta]$ and study $T = L_A$, hence $[T] = A$ and:

$$\begin{aligned}
[T]_{\beta,\beta} &= \big[[T(V_1)]_\beta \,|\, [T(V_2)]_\beta \,|\cdots|\, [T(V_n)]_\beta \big] \\
&= \big[[AV_1]_\beta \,|\, [AV_2]_\beta \,|\cdots|\, [AV_n]_\beta \big] \\
&= \big[[\lambda_1 V_1]_\beta \,|\, [\lambda_2 V_2]_\beta \,|\cdots|\, [\lambda_n V_n]_\beta \big] \\
&= \big[\lambda_1 e_1 \,|\, \lambda_2 e_2 \,|\cdots|\, \lambda_n e_n \big] \\
&= \text{Diag}(\lambda_1, \lambda_2, \dots, \lambda_n) = [\beta]^{-1} A [\beta].
\end{aligned}$$

This proves $\Rightarrow$ we identify $[\beta] = P$ and $P^{-1}AP = D$ where $D = \text{Diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$.

Proof Continued

$\Leftarrow$ Suppose there exists invertible P for which $P^{-1}AP = D$ where $D = [d_1e_1 | d_2e_2 | \cdots | d_ne_n]$ is a diagonal matrix. Let $v_1, v_2, \dots, v_n$ denote the columns of $P = [v_1 | v_2 | \cdots | v_n]$. Observe

$$
\begin{aligned}
P^{-1}AP = D &\Rightarrow AP = PD \\
&\Rightarrow A[v_1 | v_2 | \cdots | v_n] = [P d_1e_1 | P d_2e_2 | \cdots | P d_n e_n] \\
&\Rightarrow [Av_1 | \cdots | Av_n] = [d_1 Pe_1 | \cdots | d_n P e_n] \\
&\Rightarrow [Av_1 | \cdots | Av_n] = [d_1 v_1 | \cdots | d_n v_n] \\
&\Rightarrow Av_j = d_j v_j \text{ for } j = 1, 2, \dots, n.
\end{aligned}
$$

We find $\beta = \{v_1, v_2, \dots, v_n\}$ is an eigenbasis where $\lambda_1 = d_1, \lambda_2 = d_2, \dots, \lambda_n = d_n$. //

When we have a diagonalization of A then we can calculate with D in the place of A.

OBSERVATION:

If $P^{-1}AP = D$ then $A = PDP^{-1}$ and

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$$

$$A^3 = (PDP^{-1})(PDP^{-1})(PDP^{-1}) = PD^3P^{-1}$$

$$\boxed{A^n = PD^nP^{-1}}$$

note, $D^n = \begin{bmatrix} d_1^n & & & \\ & d_2^n & & \\ & & \ddots & \\ & & & d_n^n \end{bmatrix}$

exponentiating diagonal matrix easy

E1 Suppose $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Calculate A^n for $n \in \mathbb{N}$

$$\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = (\lambda-1)^2 - 4 = (\lambda+1)(\lambda-3)$$

Thus $\lambda_1 = -1$ and $\lambda_2 = 3$

($\lambda_1 + \lambda_2 = 2 = \text{trace}(A)$
$\lambda_1\lambda_2 = -1(3) = -3 = \det(A)$
couple quick checks on my algebra)

$\lambda_1 = -1$

$A + I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ $\therefore$ $V_1 = (1, -1)$ is e-vector for A and $\mathcal{E}_1 = \text{span}\{V_1\}$

$\lambda_2 = 3$

$A - 3I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$ $\therefore$ $V_2 = (1, 1)$ is e-vector for A
$\mathcal{E}_2 = \text{span}\{V_2\}$

To calculate A^n we should use $A^n = PD^nP^{-1}$ and since $V_1 \cdot V_2 = 0$ we might as well use

$\beta = \left\{ \frac{1}{\sqrt{2}}(1,-1), \frac{1}{\sqrt{2}}(1,1) \right\}$ for our eigenbasis

Then $[\beta] = P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ whereas $[\beta]^T = P^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

and since $D = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & \\ & 3 \end{bmatrix}$ it follows,

$$A^n = PD^nP^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} (-1)^n & 0 \\ 0 & 3^n \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\left[\begin{array}{c|c} (-1)^n & (-1)^{n+1} \\ \hline 3^n & 3^n \end{array}\right]$$

$$= \boxed{\frac{1}{2}\left[\begin{array}{c|c} (-1)^n + 3^n & (-1)^{n+1} + 3^n \\ \hline (-1)^{n+1} + 3^n & (-1)^n + 3^n \end{array}\right]}$$

Remark: a nice Thm we discuss in a future Lecture is that $A^T = A$ implies $\exists$ an orthonormal eigenbasis. We see that result at play here.

The calculation in E1 is typical. We use the diagonalized matrix to calculate the desired task then use $P^{-1}AP = D$ or $A = PDP^{-1}$ to derive how the calculation goes for A itself. Let's return once more to the theory and calculation of e-vectors.

E2 $A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ and note $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are vectors of note. Find eigenbasis for A

$$Av_1 = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \therefore \ v_1 \text{ is e-vector with } \lambda_1 = 5.$$

$$Av_2 = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \therefore \ v_2 \text{ is e-vector with } \lambda_2 = 1.$$

Observe $tr(A) = 2+1+2 = 5 = \lambda_1 + \lambda_2 + \lambda_3 = 6 + \lambda_3$ thus we deduce $\lambda_3 = -1$ (much easier than using $\det(A - \lambda I) = 0$ here!)

$$A + I = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & -4 & -8 \\ 0 & -2 & -4 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \text{rref}(A+I)$$

Consequently, $v_3 = (1, -2, 1) \in \text{Null}(A+I)$.

$\therefore \ \beta = \{(1,1,1), (1,0,-1), (1,-2,1)\}$ is an <u>orthogonal</u> eigenbasis for A.

<u>Remark</u>: not every eigenvalue problem requires explicit calculation of $\det(A - \lambda I)$ from A. Notice, $\underline{\det(A - \lambda I) = -(\lambda - 5)(\lambda - 1)(\lambda + 1)}$ here.

$\boxed{E3}$ $A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 2 & 0 & 0 \\ 2 & -\lambda & 3 & 0 \\ 0 & 3 & -\lambda & 2 \\ 0 & 0 & 2 & -\lambda \end{bmatrix}$$

$$= -\lambda \det \begin{bmatrix} -\lambda & 3 & 0 \\ 3 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{bmatrix} - 2 \det \begin{bmatrix} 2 & 3 & 0 \\ 0 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{bmatrix}$$

$$= -\lambda\left[-\lambda(\lambda^2 - 4) - 3(-3\lambda)\right] - 2\left[(2)(\lambda^2 - 4)\right]$$

$$= \lambda^2(\lambda^2 - 4) - 9\lambda^2 - 4\lambda^2 + 16$$

$$= \lambda^4 - 17\lambda^2 + 16$$

$$= (\lambda^2 - 16)(\lambda^2 - 1)$$

$$= (\lambda + 4)(\lambda - 4)(\lambda + 1)(\lambda - 1)$$

We find $\lambda_1 = -4,\ \lambda_2 = 4,\ \lambda_3 = -1,\ \lambda_4 = 1$.

with $V_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix},\ V_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix},\ V_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix},\ V_4 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix}$

You can check, $AV_1 = -4V_1$
$AV_2 = 4V_2$
$AV_3 = -V_3$
$AV_4 = V_4$

$\{V_1, V_2, V_3, V_4\}$ is orthogonal eigenbasis for A.

$\boxed{E4}$ $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 1 & -1 & 3-\lambda \end{bmatrix}$$

$$= (3-\lambda)\det\begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}$$

$$= -(\lambda-3)(\lambda-2)^2 \Rightarrow \underbrace{\begin{matrix}\lambda_1 = 3, & \lambda_2 = 2 \\ n_1 = 1 & n_2 = 2\end{matrix}}_{\text{algebraic multiplicity.}}$$

$\underline{\lambda_1 = 3}$ $A - 3I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore$ $\mathrm{Null}(A - 3I) = \mathrm{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

$\underline{\dim(\mathcal{E}_1) = 1 = \gamma_1 = n_1}$

$\underline{\lambda_2 = 2}$ $A - 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x \in \mathrm{Null}(A - 2I)$ has $x_1 = -x_3$, $x_2 = 0$

$\therefore$ $\mathrm{Null}(A - 2I) = \mathrm{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

$\underline{\dim(\mathcal{E}_2) = 1 = \gamma_2 \neq n_2 = 2}$

$\boxed{E5}$ $A = \begin{bmatrix} -2 & 0 & 0 \\ 4 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$ has $\det(A - \lambda I) = -(\lambda + 2)^3$

thus $n_1 = 3$ for algebraic multiplicity of $\lambda_1 = -2$.

$A + 2I = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow \mathcal{E}_1 = \mathrm{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

$\therefore$ $\gamma_1 = 2 \neq 3 = n_1$.

$\boxed{E6}$ $A = \begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 1 \\ 9 & 3 & -4 \end{bmatrix}$ has $\det(A - \lambda I) = -(\lambda+1)^3$

thus $\lambda_1 = -1$ with algebraic multiplicity $n_1 = 3$

$$A + I = \begin{bmatrix} 3 & 1 & -1 \\ -3 & 0 & 1 \\ 9 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, $\text{Null}(A+I) = \text{span}\{(1,0,3)\}$

and we find $\underline{\gamma_1 = \dim(\mathcal{E}_1) = 1 \neq 3 = n_1}$.

<u>Remark</u> : E4, E5, E6 all illustrate matrices where the algebraic and geometric multiplicities differ.

Let's conclude with a few interesting abstract Theorems about eigenvectors.

> **Th^m/** Given $A \in \mathbb{R}^{n \times n}$ and eigenvectors V_1 and V_2 with distinct eigenvalues λ_1, λ_2 we have $\{V_1, V_2\}$ is LI.

<u>Proof</u>: Suppose $AV_1 = \lambda_1 V_1$ and $AV_2 = \lambda_2 V_2$ where $\lambda_1 \neq \lambda_2$ and $V_1, V_2 \neq 0$. Consider $c_1 V_1 + c_2 V_2 = 0$ and multiply by $A - \lambda_2 I$, we obtain

$$c_1(A - \lambda_2 I)V_1 + c_2(A - \lambda_2 I)V_2 = 0$$

$$\Rightarrow c_1(AV_1 - \lambda_2 V_1) + c_2(AV_2 - \lambda_2 V_2) = 0$$

$$\Rightarrow c_1(\lambda_1 - \lambda_2)V_1 + c_2(\lambda_2 V_2 - \lambda_2 V_2) = 0$$

Hence $c_1(\lambda_1 - \lambda_2)V_1 = 0 \Rightarrow c_1 = 0$ since $\lambda_1 - \lambda_2 \neq 0$ and $V_1 \neq 0$.

Proof continued,

we had $c_1 v_1 + c_2 v_2 = 0$ and we found $c_1 = 0$.

Thus $c_2 v_2 = 0$ where $v_2 \neq 0$ and it follows $c_2 = 0$.

We've shown $c_1 v_1 + c_2 v_2 = 0 \Rightarrow c_1 = 0$ and $c_2 = 0$

Therefore, $\{v_1, v_2\}$ is LI. //

$$\boxed{\text{Th}^{\underline{m}}/\ \text{If } A \in \mathbb{R}^{n\times n} \text{ and } v_1, v_2, \dots, v_k \text{ are e-vectors with distinct e-values then } \{v_1, \dots, v_k\} \text{ is LI}}$$

Proof: Suppose $Av_i = \lambda_i v_i$ for $i = 1, 2, \dots, k$ where $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Suppose that $c_1, c_2, \dots, c_k \in \mathbb{R}$ and

$$c_1 v_1 + \cdots + c_k v_k = \sum_{i=1}^{k} c_i v_i = 0 \qquad (*)$$

Multiply by M_i where

$$M_i = (A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_{i-1} I)(A - \lambda_{i+1} I) \cdots (A - \lambda_k I)$$

and notice $M_i v_j = 0$ for $j \neq i$ (exercise for reader)

Therefore (*) yields $c_i M_i v_i = 0$ hence

$$c_i \underbrace{(\lambda_i - \lambda_1)(\lambda_i - \lambda_2) \cdots \widehat{(\lambda_i - \lambda_i)} \cdots (\lambda_i - \lambda_k)}_{\text{nonzero constant since } \lambda_i\text{-term omitted.}} v_i = 0$$

Hence $c_i = 0$ for arbitrary i and we find $c_1 = 0, c_2 = 0, \dots, c_k = 0 \therefore \{v_1, v_2, \dots, v_k\}$ LI. //