Lecture 26 : Generalized Eigenvectors & The Jordan Form

Given an $n \times n$ real matrix with real eigenvalues it is always possible to similarity transform the matrix into Jordan Form. This generalizes the project of diagonalization since the Jordan Form is a particularly nice type of matrix for calculations. For example, we'll soon see how to solve $\frac{dx}{dt} = Ax$ with the help of the Jordan Form.

Def^{n}/ We denote the $(k \times k)$ - Jordan block with eigenvalue λ by $J_k(\lambda)$. It is defined by,

$$J_1(\lambda) = \lambda, \quad J_2(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad J_3(\lambda) = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

and generally $J_k(\lambda) = \lambda I + N_k$ where N_k is the $(k \times k)$-matrix which is everywhere zero except for entries of 1 along the superdiagonal containing entries $(1,2), (2,3), (3,4), \dots, (k-1, k)$. In particular

$$N_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad N_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{E1}$ Let's look at Eigenspace and characteristic eq^{n} for a Jordan Block matrix,

$$J_4(7) = \begin{bmatrix} 7 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix} \qquad \det(J_4(7) - \lambda I) = \det \begin{bmatrix} 7-\lambda & 1 & 0 & 0 \\ & 7-\lambda & 1 & 0 \\ & & 7-\lambda & 1 \\ & & & 7-\lambda \end{bmatrix}$$

$$= (\lambda - 7)^4 \quad (n_1 = 4)$$

$$J_4(7) - 7I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \underbrace{E_{\lambda = 7} = \mathrm{span}\{(1,0,0,0)\}}_{\nu_1 = 4}$$

Generally the Jordan Form is comprised of a $\oplus$ of Jordan Blocks.

Def^n/ If $A \in \mathbb{R}^{n\times n}$ and there exists a basis β for which $[\beta]^{-1}A[\beta] = J_{k_1}(\lambda_1) \oplus J_{k_2}(\lambda_2) \oplus \cdots \oplus J_{k_s}(\lambda_s) = J_A$ then J_A as above is a Jordan form of A with s-Jordan blocks. We say β is a <u>Jordan Basis</u> in this case.

We'll see that a matrix in Jordan form like J_A above has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_s$ (these don't have to be distinct) and we obtain just one e-vector up to LI for each block.

$\boxed{E2}$ $A = J_2(3) \oplus J_2(3) \oplus J_3(2)$

$$A - \lambda I = (J_2(3) - \lambda I_2) \oplus (J_2(3) - \lambda I_2) \oplus (J_3(2) - \lambda I_3)$$

$$\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix} \det\begin{pmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix} \det\begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

$$= (\lambda-3)^4(2-\lambda)^3$$

$\lambda_1 = 3,\ n_1 = 4$ and $\lambda_2 = 2,\ n_2 = 3$

$$A = \left[\begin{array}{cc|cc|ccc} 3 & 1 & & & & & \\ 0 & 3 & & & & & \\ \hline & & 3 & 1 & & & \\ & & 0 & 3 & & & \\ \hline & & & & 2 & 1 & 0 \\ & & & & 0 & 2 & 1 \\ & & & & 0 & 0 & 2 \end{array}\right]$$

$$\underbrace{\begin{aligned} Ae_1 &= 3e_1 \\ Ae_3 &= 3e_3 \\ Ae_5 &= 2e_5 \end{aligned}}_{e_1,\, e_3,\, e_5 \text{ are eigenvectors for } A}$$

$\mathcal{E}_{\lambda_1} = \operatorname{span}\{e_1, e_3\},\ \nu_1 = 2$

$\mathcal{E}_{\lambda_2} = \operatorname{span}\{e_5\},\ \nu_2 = 1$

Let's detail the structure of a Jordan Basis,

Def/ Suppose $v_1, v_2, \dots, v_k$ are nonzero vectors for which $(A-\lambda I)v_1 = 0$ and $(A-\lambda I)v_j = v_{j-1}$ for $j = 1, 2, \dots, k$ then $v_1, v_2, \dots, v_k$ forms a <u>k-chain</u> for A with eigenvalue λ.

Then we can define Jordan Basis,

Def/ β is a <u>Jordan Basis</u> for A if β is formed by the union of k-chains to give β LI set of n-vectors (assuming A is $n \times n$)

Let's see why the chain-conditions put the matrix into Jordan form,

Th/ Let $\beta = \{v_1, \dots, v_k\}$ be k-chain with eigenvalue λ for A then $T = L_A : \mathbb{R}^k \to \mathbb{R}^k$ has $[T]_{\beta,\beta} = J_k(\lambda)$

<u>Proof</u>: Suppose $(A-\lambda I)v_1 = 0$ and $(A-\lambda I)v_j = v_{j-1}$ for $j = 1, 2, \dots, k$ then $T(v_1) = Av_1 = \lambda v_1$ and since $Av_j = \lambda v_j + v_{j-1}$ we have $T = L_A$ with $T(v_j) = \lambda v_j + v_{j-1}$. Thus,

$$
\begin{aligned}
[T]_{\beta,\beta} &= [[T(v_1)]_\beta \,|\, [T(v_2)]_\beta \,|\, \cdots \,|\, [T(v_k)]_\beta] \\
&= [\,[\lambda v_1]_\beta \,|\, [\lambda v_2 + v_1]_\beta \,|\, \cdots \,|\, [\lambda v_k + v_{k-1}]_\beta\,] \\
&= [\,\lambda e_1 \,|\, \lambda e_2 + e_1 \,|\, \cdots \,|\, \lambda e_k + e_{k-1}\,] \\
&= \lambda [e_1 | e_2 | \cdots | e_k] + [0 | e_1 | \cdots | e_{k-1}] \\
&= \lambda I_k + N_k \\
&= J_k(\lambda),
\end{aligned}
$$

where $\beta = \{v_1, \dots, v_k\}$. //

Corollary: for A and β as in the Th^m, $[\beta]^{-1} A [\beta] = J_k(\lambda)$

The previous Th^m studied a single k-chain, typically we need multiple chains to form the Jordan Basis, one chain for each block. The proof of the following is beyond this course,

> **Th^m/** Given $A \in \mathbb{R}^{n\times n}$ such that all zeros of $\det(A-\lambda I)=0$ are real then there exists a collection of k_i-chains β_i with e-values λ_i such that $\beta_1 \cup \beta_2 \cup \cdots \cup \beta_s$ forms basis for $\mathbb{R}^n$ and $[\beta]^{-1}A[\beta] = J_{k_1}(\lambda_1)\oplus J_{k_2}(\lambda_2)\oplus\cdots\oplus J_{k_s}(\lambda_s)$

When a matrix is a Jordan form, the standard basis is Jordan basis.

$\boxed{E3}$

$$A = \left[\begin{array}{ccc|cccc|cc} 3 & 1 & 0 & & & & & & \\ 0 & 3 & 1 & & & & & & \\ 0 & 0 & 3 & & & & & & \\ \hline & & & -4 & 1 & 0 & 0 & & \\ & & & & -4 & 1 & 0 & & \\ & & & & & -4 & 1 & & \\ & & & & & & -4 & & \\ \hline & & & & & & & 7 & 1 \\ & & & & & & & 0 & 7 \end{array}\right] = J_3(3)\oplus J_4(-4)\oplus J_2(7)$$

$$\left\{\begin{array}{l} (A-3I)e_1 = 0 \\ (A-3I)e_2 = e_1 \\ (A-3I)e_3 = e_2 \end{array}\right\}$$

3-chain e_1, e_2, e_3 with $\lambda_1 = 3$

$$\left\{\begin{array}{l} (A+4I)e_4 = 0 \\ (A+4I)e_5 = e_4 \\ (A+4I)e_6 = e_5 \\ (A+4I)e_7 = e_6 \end{array}\right\}$$

4-chain e_4, e_5, e_6, e_7 with $\lambda_2 = -4$

$$\left\{\begin{array}{l} (A-7I)e_8 = 0 \\ (A-7I)e_9 = 0 \end{array}\right\}$$

2-chain e_8, e_9 with $\lambda_3 = 7$

THE MATRIX EXPONENTIAL

We now return to the problem of solving $\frac{dx}{dt} = Ax$. The theory of DEq$^{\text{n}}$'s tells us the general solution is formed by $x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$ where $\frac{dx_j}{dt} = Ax_j$ for $j = 1, 2, \dots, n$. In short, we need to find n-LI fundamental solutions to form the general sol$^{\text{n}}$. When A is diagonalizable then $x_j = e^{t\lambda_j} V_j$ for $j = 1, 2, \dots, n$ are LI solutions if $V_1, V_2, \dots, V_n$ is LI (eigenbasis for A) Now we face the case A not diagonalizable, yet A has <u>real</u> <u>eigenvalues</u>. The series below <u>always</u> converges,

$$\boxed{\text{Defn}/\ e^{tA} = I + tA + \frac{1}{2}t^2A^2 + \frac{1}{3!}t^3A^3 + \cdots = \sum_{n=0}^{\infty} \frac{t^n}{n!}A^n}$$

The direct calculation of e^{tA} is challenging, we'll look at that more next Lecture. Our use of it here is a bit indirect...

Thm/
1.) If $AB = BA$ then $e^A e^B = e^{A+B}$,
2.) $(e^A)^{-1} = e^{-A}$ hence $\det(e^{tA}) \neq 0$,
3.) $\frac{d}{dt}(e^{tA}) = Ae^{tA}$

<u>Proof</u>: I'll focus on 3.)

$$\begin{aligned}
\frac{d}{dt}\left(e^{tA}\right) &= \frac{d}{dt}\left[I + tA + \frac{1}{2}t^2A^2 + \frac{1}{3!}t^3A^3 + \cdots \right] \\
&= A + tA^2 + \frac{1}{2}t^2A^3 + \cdots \\
&= A(I + tA + \frac{1}{2}t^2A^2 + \cdots) \\
&= Ae^{tA}
\end{aligned}$$

THE MAGIC FORMULA

Notice $MN = NM$ implies $e^M e^N = e^{M+N}$. Think of $M = \lambda I t$ and $N = (A - \lambda I)t$ in what follows,

$$\begin{aligned} e^{tA} &= e^{t(A - \lambda I + \lambda I)} \\ &= e^{\lambda I t + t(A - \lambda I)} \\ &= e^{\lambda I t} e^{t(A - \lambda I)} \end{aligned}$$

$$\boxed{e^{tA} = e^{\lambda t}\left(I + t(A - \lambda I) + \frac{t^2}{2}(A - \lambda I)^2 + \frac{t^3}{3!}(A - \lambda I)^3 + \cdots\right)}$$

Suppose $v_1, v_2, \dots, v_k$ is k-chain for A with e-value λ then observe the chain-eq^{ns} $(A - \lambda I)v_1 = 0$ and $(A - \lambda I)v_i = v_{i-1}$ yield the identities below:

1.) $(A - \lambda I)v_1 = 0$

2.) $(A - \lambda I)v_2 = v_1$, $(A - \lambda I)^2 v_2 = 0$

3.) $(A - \lambda I)v_3 = v_2$, $(A - \lambda I)^2 v_3 = v_1$, $(A - \lambda I)^3 v_3 = 0$

4.) $(A - \lambda I)^j v_i = v_{i-j}$ for $j \leq i - 1$ and
$(A - \lambda I)^j v_i = 0$ for $j \geq i$

Now use the identities above to simplify $e^{tA} v_j$ in view of the magic formula boxed in green,

Th^m/ Given $v_1, v_2, \dots, v_k$ a k-chain with e-value λ,

1.) $e^{tA} v_1 = e^{\lambda t} v_1$

2.) $e^{tA} v_2 = e^{\lambda t}(v_2 + t v_1)$

3.) $e^{tA} v_3 = e^{\lambda t}(v_3 + t v_2 + \frac{1}{2} t^2 v_1)$

4.) $e^{tA} v_j = e^{\lambda t}\left(v_j + t v_{j-1} + \cdots + \frac{1}{(j-1)!} t^{j-1} v_1\right)$

E4 Let $A = J_3(8) = \begin{bmatrix} 8 & 1 & 0 \\ 0 & 8 & 1 \\ 0 & 0 & 8 \end{bmatrix}$ then we have

$A - 8I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and clearly $(A-8I)e_1 = 0$, $(A-8I)e_2 = e_1$, $(A-8I)e_3 = e_2$

Hence e_1, e_2, e_3 is 3-chain with $\lambda = 8$ for A and hence the differential eq^n $\frac{dx}{dt} = Ax$,

$$\boxed{x = c_1 e^{8t} e_1 + c_2 e^{8t}(e_2 + te_1) + c_3 e^{8t}\left(e_3 + te_2 + \tfrac{1}{2}t^2 e_1\right)}$$

E5 $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$ has $\lambda_1 = 3$ and $\lambda_2 = 2$ and we saw in **E4** of Lecture 25 that $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ were e-vectors with e-values 3 and 2 respective. However, the algebraic multiplicity for $\lambda_2 = 2$ was $n_2 = 2$. Let's find the <u>generalized</u> <u>eigenvector</u> of <u>order 2</u> to make v_2, v_3 a 2-chain for $\lambda_2 = 2$,

$$(A - 2I)v_3 = v_2 \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{matrix} \longrightarrow v = -1 \\ \\ \longrightarrow u - v + w = 0 \end{matrix}$$

$u = v - w = -1 - w$

select $w = 0$

$\therefore\ v_3 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$.

Now we solve $\frac{dx}{dt} = Ax$,

$$e^{tA}v_1 = e^{3t}v_1, \quad e^{tA}v_2 = e^{2t}v_2, \quad e^{tA}v_3 = e^{2t}(v_3 + tv_2)$$

Thus the general solution is

$$\boxed{x = c_1 e^{3t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{2t}\left(\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)}$$

$\boxed{E6}$ In $\boxed{E6}$ of LECTURE 25 we found $A = \begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 1 \\ 9 & 3 & -4 \end{bmatrix}$ has $\det(A - \lambda I) = -(\lambda+1)^3$ hence $\lambda_1 = -1$ with $n_1 = 3$ yet $\text{Null}(A+I) = \text{span}\{(1,0,3)\}$. Let $v_1 = (1,0,3)$ we wish to calculate v_2 and v_3 which solve

(1) $(A+I)v_2 = v_1$ and (2) $(A+I)v_3 = v_2$.

(1) $(A+I)v_2 = v_1 \longrightarrow \underline{\begin{bmatrix} 3 & 1 & -1 \\ -3 & 0 & 1 \\ 9 & 3 & -3 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}}$ *

(row 1) + (row 2) : $v = 1$

Then $-3u + w = 0 \rightarrow w = 3u$, can choose $u = 0$

we find solution $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (which is actually really obvious if you look back at * again)

(2) $(A+I)v_3 = v_2 \rightarrow \begin{bmatrix} 3 & 1 & -1 \\ -3 & 0 & 1 \\ 9 & 3 & -3 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

You can calculate $\text{rref}\left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ -3 & 0 & 1 & 1 \\ 9 & 3 & -3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -1/3 & -1/3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Then select $v_3 = (\frac{-1}{3}, 1, 0)$ by setting $w = 0$.

In summary, $\underline{(1,0,3), (0,1,0), (-1/3, 1, 0)}$ is $\lambda_1 = -1$ 3-chain

It follows $\boxed{x = c_1 e^{-t}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + c_2 e^{-t}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}\right) + c_2 e^{-t}\left(\begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}\right)}$

Remark: $\boxed{E5}$ of LECTURE 25 studied $A = \begin{bmatrix} -2 & 0 & 0 \\ 4 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$ with $\lambda_1 = -2$, $n_1 = 3$ and there $\nu_1 = 2$. The analog of E6's calculation is more troublesome...

GENERALIZED EIGENVECTORS

I suppose this should have gone earlier, but better late than never.

Def^{n}/ Given $A \in \mathbb{R}^{n\times n}$, if $\det(A-\lambda I)=0$ for some $\lambda \in \mathbb{R}$ then we say $v \in \mathbb{R}^n$ for which $(A-\lambda I)^{p-1}v \neq 0$ yet $(A-\lambda I)^p v = 0$ is a <u>generalized</u> <u>e-vector</u> of <u>order p</u>.

If we return to our calculations we can see the following is true:

Th^{m}/ Given k-chain $v_1, v_2, \dots, v_k$ for eigenvalue λ we have v_j is a generalized e-vector of order j for $j = 1, 2, \dots, k$

$$(A-\lambda I)v_1 = 0 \quad , \quad (A-\lambda I)^0 v_1 = Iv_1 = v_1 \neq 0$$

$$(A-\lambda I)^2 v_2 = 0 \quad , \quad (A-\lambda I)v_2 = v_1 \neq 0$$

$$(A-\lambda I)^3 v_3 = 0 \quad , \quad (A-\lambda I)^2 v_3 = v_1 \neq 0$$

$$\vdots$$

$$(A-\lambda I)^k v_k = 0 \quad , \quad (A-\lambda I)^{k-1} v_k = v_1 \neq 0.$$