Lecture 27: Matrix Exponential & Cayley Hamilton $Th^{\underline{m}}$

Our application of interest is the solution of $\frac{dx}{dt} = Ax$ in the case A is an $n\times n$ constant matrix. A solution $x : \mathbb{R} \longrightarrow \mathbb{R}^n$ is a vector of functions of t. However, it is nice to think about a bunch of solutions at the same time, last time we argued the matrix exponential was such an object. Let's work a bit more on the theory here.

> **$Th^{\underline{m}}$/** Suppose $\Xi = [x_1 | x_2 | \cdots | x_n]$ is a matrix of functions of t. Then
> $$\frac{d\Xi}{dt} = A\Xi \iff \frac{d}{dt}\left(\text{col}_j(\Xi)\right) = A\,\text{col}_j(\Xi) \quad \text{for } j = 1, 2, .., n$$

Proof: Let $\Xi = [x_1 | x_2 | \cdots | x_n]$ and $A \in \mathbb{R}^{n\times n}$. Observe,

$$\frac{d\Xi}{dt} = A\Xi \iff \left[\frac{dx_1}{dt} \middle| \frac{dx_2}{dt} \middle| \cdots \middle| \frac{dx_n}{dt}\right] = A\left[x_1 | x_2 | .. | x_n\right]$$

$$\iff \left[\frac{dx_1}{dt} \middle| \frac{dx_2}{dt} \middle| .. \middle| \frac{dx_n}{dt}\right] = \left[Ax_1 | Ax_2 | \cdots | Ax_n\right]$$

$$\iff \frac{dx_j}{dt} = Ax_j \quad \text{for } j = 1, 2, .., n. //$$

- This $Th^{\underline{m}}$ says Ξ is a solution matrix if and only if each column of Ξ is itself a solution.

Last class we saw $\frac{d}{dt}\left(e^{tA}\right) = Ae^{tA}$ hence,

> **$Th^{\underline{m}}$/** $\frac{dx}{dt} = Ax$ has solution matrix e^{tA}.

The theory of analysis for ODE's shows <u>any</u> solution of $\frac{dx}{dt} = Ax$ can be expressed in terms of a given <u>invertible solution matrix</u> and a vector of constants. Since $\det(e^{tA}) \neq 0$,

Thm/ The general solution of $\frac{dx}{dt} = Ax$ is given by $x = e^{tA}c$ where $c \in \mathbb{R}^n$ is constant.

The above result means if we can calculate e^{tA} then solving $\frac{dx}{dt} = Ax$ is very easy, we just multiply e^{tA} by $c = (c_1, c_2, \dots, c_n)$ and write it down.

$\boxed{E1}$ $N_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $N_3^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $N_3^3 = 0$

$$e^{tN_3} = I + tN_3 + \tfrac{1}{2}t^2N_3^2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \tfrac{1}{2}t^2\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $\frac{dx}{dt} = N_3 x$ has sol^{n} $x = e^{tN_3}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$$\boxed{x = c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} t^2/2 \\ t \\ 1 \end{bmatrix}}$$

Example 1 was particularly easy, let's try something a bit more troublesome,

E2 $A = J_3(7) = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{bmatrix} = 7I + N_3$

$$\begin{aligned} e^{tA} &= e^{t(A - 7I) + 7It} \\ &= e^{7t} e^{(A-7I)t} \\ &= e^{7t} e^{tN_3} \\ &= e^{7t} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$\therefore \dfrac{dx}{dt} = Ax$ has sol^n

$$\boxed{x = c_1 e^{7t}\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 e^{7t}\begin{bmatrix}t\\1\\0\end{bmatrix} + c_3 e^{7t}\begin{bmatrix}t^2/2\\t\\1\end{bmatrix}}$$

Continuing down this path,

$$\boxed{\text{Th}^m/\ \exp(t J_n(\lambda)) = e^{\lambda t}\left(I + tN_n + \tfrac{1}{2}t^2 N_n^2 + \cdots + \frac{t^n}{(n-1)!}N_n^{n-1}\right)}$$

E3 $\exp\left(t J_4(1)\right) = e^t \begin{bmatrix} 1 & t & t^2/2 & t^3/6 \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Th^m/ CAYLEY HAMILTON

Given A an $n \times n$ matrix, if

$$\det(xI - A) = x^n + c_{n-1}x^{n-1} + \cdots + c_2x^2 + c_1x + c_0$$

then $A^n + c_{n-1}A^{n-1} + \cdots + c_2A^2 + c_1A + c_0I = 0$

In other words if $P_A(x)$ is the characteristic polynomial for A then A solves its own characteristic equation $P_A(x) = 0$. This also holds if we use $\det(A - xI)$ for the characteristic poly. of A.