# Lecture 28: Properties and Examples of Complex Eigenvectors

I'll begin with a somewhat relevant matrix exponential calculation,

$\boxed{E1}$ Let $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ then $J^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$ and $J^3 = -J$ whereas $J^4 = (-I)(-I) = I$ and thus,

$$
\begin{aligned}
e^{tJ} &= I + tJ + \tfrac{1}{2}t^2J^2 + \tfrac{1}{3!}t^3J^3 + \tfrac{1}{4!}t^4J^4 + \cdots \\
&= I\left(1 - \tfrac{1}{2}t^2 + \tfrac{1}{4!}t^4 + \cdots\right) + J\left(t - \tfrac{1}{3!}t^3 + \tfrac{1}{5!}t^5 + \cdots\right) \\
&= \cos t\, I + \sin t\, J \\
&= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}
\end{aligned}
$$

Notice the functions above we could not have produced with our real e-value/vector solutions. This brings us to:

**Def$^{n}$/** Let $A \in \mathbb{R}^{n \times n}$ then if $\lambda = \alpha + i\beta$ with $\beta \neq 0$ is a solution of $\det(A - \lambda I) = 0$ then $\lambda$ is a <u>complex eigenvalue</u> of $A$. If $v = a + ib$ with $b \neq 0$ and $a, b \in \mathbb{R}^n$ satisfy

$$Av = \lambda v$$

then $v$ is complex eigenvector with e-value $\lambda$ for $A$

I'll forego the proof, but whenever $\lambda$ is complex eigenvalue there is always at least one LI complex e-vector and in fact $\{a, b\}$ is LI over $\mathbb{R}$ where $v = a + ib$ is the complex e-vector.

$\boxed{E2}$ $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\det(J - \lambda I) = \det\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$\underline{\lambda_1 = i}$ $\quad J - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} -1 \\ i \end{bmatrix}$

$\underline{\lambda_2 = -i}$ $\quad J + iI = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} -1 \\ -i \end{bmatrix}$

Def$^n$/ $(\alpha + i\beta)^* = \alpha - i\beta$ and $(a+ib)^* = a - ib$ define the <u>complex</u> <u>conjugates</u>

If the matrix $A$ is real then $\det(A - \lambda I)$ is a polynomial with real coefficients. It follows that whenever $\lambda = \alpha + i\beta$ is a complex eigenvalue then $\lambda^* = \alpha - i\beta$ is also a complex eigenvalue of $A$. Observe,

$$Av = \lambda v \Rightarrow (Av)^* = (\lambda v)^*$$
$$\Rightarrow \underbrace{Av^* = \lambda^* v^*}_{\substack{v^* \text{ is e-vector} \\ \text{with e-value } \lambda^*}}$$

$$\begin{matrix} v = a + ib \\ \lambda = \alpha + i\beta \end{matrix} \quad \Longrightarrow \quad \begin{matrix} v^* = a - ib \\ \lambda^* = \alpha - i\beta \end{matrix}$$