Lecture 29: Generalized Complex Eigenvectors...

We saw last class that complex eigenvectors for $A \in \mathbb{R}^{n\times n}$ come in conjugate pairs. For a 2x2 matrix A with $Av = \lambda v$ where $v = a+ib$ and $\lambda = \alpha + i\beta$ we also have $Av^* = \lambda^* v^*$ and $v^* = a - ib$, $\lambda^* = \alpha - i\beta$. Such a matrix can be

(1.) complex diagonalized by $P = [v | v^*]$ to similarity transform to $P^{-1}AP = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^* \end{bmatrix}$

(2.) transformed to <u>real</u> <u>Jordan</u> <u>form</u> via $\gamma = [a | b]$ for which $[\gamma]^{-1}A[\gamma] = \underbrace{\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}}_{R_2(\alpha+i\beta)}$

We should discuss how to construct the <u>real</u> solution to $\frac{dx}{dt} = Ax$ in the case A has $\lambda = \alpha \pm i\beta$ as above.

Last time we calculated $e^{tJ} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

so since e^{tJ} is solution matrix for $\frac{dx}{dt} = Jx$ we should expect sines and cosines. We need the following identity to unravel this problem,

$$\boxed{\underline{\text{Euler's Identity:}}\ e^{i\theta} = \cos\theta + i\sin\theta}$$

Then calculate
$$\begin{aligned} e^{(\alpha+i\beta)t} &= e^{\alpha t + i\beta t} \\ &= e^{\alpha t} e^{i\beta t} \\ &= e^{\alpha t}(\cos\beta t + i\sin\beta t) \end{aligned}$$

COMPLEX SOLUTION $\Rightarrow$ PAIR OF REAL SOLUTIONS

If $z = x + iy$ then $\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$

thus whenever $\frac{dz}{dt} = Az$ we find

$$\underline{\underline{\frac{dx}{dt}}} + i\underline{\underline{\frac{dy}{dt}}} = A(x+iy) = \underline{\underline{Ax}} + i\,\underline{\underline{Ay}}$$

$$\frac{dx}{dt} = Ax \quad \text{and} \quad \frac{dy}{dt} = Ay$$

Th^m/ If $z = x+iy$ solves $\frac{dz}{dt} = Az$ then both x and y are real solutions of $\frac{d\vec{r}}{dt} = A\vec{r}$

Consider $Av = \lambda v$ where $\lambda = \alpha + i\beta$ and $v = a + ib$

$$\begin{aligned}\frac{d}{dt}\left(e^{\lambda t}v\right) &= \lambda e^{\lambda t}v \\ &= e^{\lambda t}\lambda v \\ &= e^{\lambda t}Av \\ &= A\left(e^{\lambda t}v\right)\end{aligned}$$

YES, THE CALCULUS STILL WORKS WHEN $\lambda = \alpha + i\beta$

Thus $z = e^{\lambda t}v$ is a <u>complex</u> <u>sol^n</u> of $\frac{dx}{dt} = Ax$.

$$z = e^{\alpha t}(\cos\beta t + i\sin\beta t)(a+ib)$$

$$= \underbrace{e^{\alpha t}\left((\cos\beta t)a - (\sin\beta t)b\right)}_{x} + i\,\underbrace{e^{\alpha t}\left((\sin\beta t)a + (\cos\beta t)b\right)}_{y}$$

Th^m/ If $v = a+ib$ has eig

$\boxed{\begin{array}{l}\text{Th}^{m}\text{/ Given } A \in \mathbb{R}^{n\times n} \text{ has complex eigenvector } v = a + ib \text{ with } \lambda = \alpha + i\beta \\ \text{the differential eq}^{n}\ \frac{dx}{dt} = Ax \text{ has real solutions} \\ e^{\alpha t}\big((\cos\beta t)a - (\sin\beta t)b\big) \text{ and } e^{\alpha t}\big((\sin\beta t)a + (\cos\beta t)b\big)\end{array}}$

E1 $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $\det(J - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1$

$\underline{\lambda_1 = i}$ | $(J - iI)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{array}{l} -iu + v = 0 \\ v = iu \end{array}$

$\underline{v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}}$.

$$v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{a_1} + i\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b_1}$$

We obtain solutions with $\alpha = 0$, $\beta = 1$,

$$e^{0\cdot t}\left((\cos t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin t\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right),\ e^{0\cdot t}\left(\sin t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos t\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

This means the <u>general</u> <u>real</u> <u>solution</u> to $\frac{dx}{dt} = Jx$ is given by $\boxed{x = c_1\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}$

<u>Remark</u>: this solution also could be found from $e^{tJ}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ since we calculated $e^{tJ} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$.

(this doesn't have to happen, generally, a similarity transform masks it...)

Def^n/ If $A \in \mathbb{R}^{n \times n}$ has complex eigenvalue $\lambda = \alpha + i\beta$ and there are nonzero vectors (complex) $V_1, V_2, \ldots, V_k$ for which $(A - \lambda I)V_1 = 0$ and $(A - \lambda I)V_j = V_{j-1}$ for $j = 2, \ldots, k$ then $V_1, V_2, \ldots, V_k$ is a k-chain of complex e-vectors and V_j is generalized complex e-vector of order j for each $j = 1, 2, \ldots, k$

Notice complex conjugation shows complex k-chains come in conjugate pairs

$$(A - \lambda I)V_j = V_{j-1} \Rightarrow (A - \lambda^* I)V_j^* = V_{j-1}^*$$

Given a 2-chain with $\lambda = \alpha + i\beta$

$$(A - \lambda I)V_1 = 0, \quad (A - \lambda I)V_2 = V_1$$

magic formula for e^{tA} still works for λ complex

$$\underbrace{z_1 = e^{\lambda t}V_1 \qquad z_2 = e^{\lambda t}(V_2 + tV_1)}_{\text{we can extract 4 real solutions from this pair of complex solutions.}}$$

$\frac{dx}{dt} = Ax$ has general solution, supposing $\lambda = \alpha + i\beta$ and $V_1 = a_1 + ib_1$ and $V_2 = a_2 + ib_2$

$$\boxed{\begin{aligned} x = {} & c_1 e^{\alpha t}\big((\cos\beta t)a_1 - (\sin\beta t)b_1\big) + c_2 e^{\alpha t}\big((\sin\beta t)a_1 + (\cos\beta t)b_1\big) \\ & + c_3 e^{\alpha t}\big((\cos\beta t)[a_2 + ta_1] - (\sin\beta t)[b_2 + tb_1]\big) \\ & + c_4 e^{\alpha t}\big((\sin\beta t)[a_2 + ta_1] + (\cos\beta t)[b_2 + tb_1]\big) \end{aligned}}$$

E2 $A \in \mathbb{R}^{4\times 4}$ with
$$(A-\lambda I)v_1 = 0$$
$$(A-\lambda I)v_2 = v_1$$
$$(A-\lambda^* I)v_1^* = 0$$
$$(A-\lambda^* I)v_2^* = v_1^*$$

If we use $P = [v_1 | v_2 | v_1^* | v_2^*]$ then

$$\boxed{P^{-1}AP = J_2(\lambda)\oplus J_2(\lambda^*) = \left[\begin{array}{cc|cc} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ \hline 0 & 0 & \lambda^* & 1 \\ 0 & 0 & 0 & \lambda^* \end{array}\right]}$$

However, if we instead use the real matrix $[a_1 | b_1 | a_2 | b_2] = Q$ from $v_1 = a_1 + ib_1$, $v_2 = a_2 + ib_2$, for $\lambda = \alpha + i\beta$,

$$\boxed{Q^{-1}AQ = \underbrace{\left[\begin{array}{cc|cc} \alpha & \beta & 1 & 0 \\ -\beta & \alpha & 0 & 1 \\ \hline 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{array}\right]}_{R_4(\alpha+i\beta)}}$$

REAL JORDAN 4×4 BLOCK FOR $\lambda = \alpha + i\beta$

Th^{m}/ If $A \in \mathbb{R}^{n\times n}$ then there exists a basis β formed by union of real and complex k-chains (real & imag. parts) for which

$$[\beta]^{-1}A[\beta] = J_1 \oplus J_2 \oplus \cdots \oplus J_s$$

and each J_i is either a real Jordan $J_{k_i}(\lambda)$ or $R_{k_i}(\alpha_i + i\beta_i)$

$\boxed{E3}$ $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$ can show $\lambda = 2 \pm i$, $v = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$

$\boxed{E4}$ $A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$ can show $\lambda = 2 \pm 3i$, $v = \begin{bmatrix} 1+3i \\ 2 \end{bmatrix}$

$\boxed{E5}$ $A = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$ can show $\lambda = 2 \pm 2i$, $v = \begin{bmatrix} 1 \\ 2-2i \end{bmatrix}$

$\boxed{E6}$ $A = \begin{bmatrix} -8 & -12 & -6 \\ 2 & 1 & 2 \\ 7 & 12 & 5 \end{bmatrix}$