LECTURE 2 : REDUCED ROW ECHELON FORM

Let's define the reduced echelon form,

Def^n/ A rectangular matrix is in echelon form if

(1.) any rows of zeros are below nonzero rows.

(2.) leading nonzero entries are found on the top row then the leading entries below the top rows are further right than those leading entries above that row.

(3.) all entries in a column below a leading entry are zero

Furthermore, the matrix is in reduced-row-echelon-form (rref) if the matrix is in echelon form and additionally

(4.) the leading entries in each nonzero row is 1.

(5.) each leading 1 is the <u>only</u> nonzero entry in its column.

E1

$$\begin{bmatrix} 3 & 6 & 4 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

leading entries are 3 and 2, this matrix is in ref-form.

E2

$$\begin{array}{c} \begin{bmatrix} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \quad\quad e_1 \;\; e_2 \quad\;\; e_3 \end{array}$$

leading entries are 1 and that leading 1 is only nonzero entry in that column.

Def^{n}/ The <u>standard</u> <u>basis</u> $e_1, e_2, \dots, e_n$ are column vectors where $(e_i)_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$$\mathbb{R}^2: \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbb{R}^3: \quad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbb{R}^4: \quad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

I like to understand rref in terms of the standard basis. In particular,

<u>Remark</u>: when a matrix is put into reduced row echelon form with k-nonzero rows then the columns with leading entries will be $e_1, e_2, e_3, \dots, e_k$ found from left to right where e_1 is 1^{st} nonzero column.

$\boxed{E3}$

$$\begin{array}{c} \begin{array}{ccccccccc} & & e_1 & & e_2 & & e_3 & e_4 & \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 3 & 0 & 6 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 7 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \end{array}$$

we call e_1, e_2, e_3, e_4 the <u>pivot</u> <u>columns</u>

Elementary Row Operations, Gauss-Jordan Algorithm

Our goal is to describe a general strategy to convert any given matrix A into $\text{rref}(A)$.

Def$^{\underline{n}}$/ Given a matrix there are three <u>elementary row operations</u>

(1.) the <u>row swap</u>

(2.) <u>rescale row</u> by nonzero multiple (that is multiply by nonzero #)

(3.) replace r_j with $r_j + ar_i$ (also called <u>row addition</u>

$\boxed{\text{E4}}$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = R_1$$

$$A \xrightarrow{2r_1} \begin{bmatrix} 2 & 2 & 2 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = R_2$$

$$A \xrightarrow{r_2 - 3r_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix} = R_3$$

We say $A \sim R_1$ and $A \sim R_2$ and $A \sim R_3$. These notations indicate R_1, R_2, R_3 are <u>row-equivalent to A</u>

<u>Remark:</u> if $[A|b] \sim [R_1|b_1]$ then the original system of equations with aug. coeff. matrix $[A|b]$ has the <u>same solution set</u> as the system of equations with aug. coeff. mat. $[R_1|b_1]$. That is why all 3 row ops are allowed.

Theorem: Gauss Jordan Algorithm

Given an $m \times n$ matrix A, there exists a finite sequence of elementary row operations which transforms A into $\operatorname{rref}(A)$ where $A \sim \operatorname{rref}(A)$.

In particular,

1.) use row swaps to move all zero rows to lower rows of matrix, continue to do this in any future step which produces a zero row above a nonzero row.

2.) use row swaps if necessary to move a nonzero entry to the topmost row in the leftmost nonzero column.

3.) identify the nonzero entry moved to top row from step 2 as the 1st pivot position, use row addition to zero out all entries below the pivot position.

4.) go to next column for which a nonzero entry is in the leftmost nonzero column, but not the column of the 1st pivot. Once more, use row swaps to move the 2nd nonzero entry to row 2. This gives us the 2nd pivot position.

5.) use row addition to zero out all entries <u>below</u> the 2nd pivot.

Continuing,

6.) following same procedure as in 4, find 3^{rd} pivot position in 3^{rd} row and then use row addition to zero out all entries below the 3^{rd} pivot.

7.) this continues until we have produced k-pivot positions in the top k-rows and all rows below the k^{th} row are zero.

(this completes the "forward pass")

8.) starting with the k^{th} pivot, use row addition to zero out all entries above the k^{th} pivot until the k^{th} column ~~~~ is simply e_k after rescaling.

multiply pivot row to make leading entry 1.

9.) repeat step 8.) for $(k-1)$-th pivot and $(k-2)$-th etc... until you reach the 2^{nd} and finally 1^{st} pivot. At this point your pivot columns are $e_1, e_2, \dots, e_k$ found in that order left to right where e_1 is in the leftmost nonzero column.

10.) conclude, the result from 9 is $\mathrm{rref}(A)$.

(this completes the backwards pass)

I'll try to follow the algorithm despite my fraction aversion,

(pivot positions)

$\boxed{E5}$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow[r_3 - \frac{1}{2}r_1]{r_2 - r_1} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 5/2 & 7/2 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 5/2 & 7/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{r_1 - \frac{2}{5}r_2} \begin{bmatrix} 2 & 0 & -2/5 \\ 0 & 5/2 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\frac{2}{5}r_2]{r_1/2} \begin{bmatrix} 1 & 0 & -1/5 \\ 0 & 1 & 7/5 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{rref}(A)$$

$\boxed{E6}$

$$B = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[r_3 - r_1]{r_2 - r_1} \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{r_3 - r_2} \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_2/3} \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathrm{rref}(B).$$

$\boxed{E7}$

$$M = \begin{bmatrix} 1 & 2 & 3 & 4 & 6 & 0 \\ 0 & 0 & 1 & 5 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 & 6 & 0 \end{bmatrix} \xrightarrow{r_4 - r_1} \begin{bmatrix} 1 & 2 & 3 & 4 & 6 & 0 \\ 0 & 0 & 1 & 5 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[r_2 - 7r_3]{r_1 - 6r_3} \begin{bmatrix} 1 & 2 & 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 - 3r_2} \begin{bmatrix} 1 & 2 & 0 & -11 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathrm{rref}(M)$$

E8 What condition is needed for a, b, c in order for the equations below to be consistent?

$$\begin{aligned} x - y + 3z &= a \\ 2x + 2z &= b \\ x + 2y - 3z &= c \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 2 & 0 & 2 & b \\ 1 & 2 & -3 & c \end{array}\right] \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \left[\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 0 & 2 & -4 & b-2a \\ 0 & 3 & -6 & c-a \end{array}\right]$$

$$\xrightarrow[2r_3]{3r_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 0 & 6 & -12 & 3b-6a \\ 0 & 6 & -12 & 2c-2a \end{array}\right] \xrightarrow{r_3 - r_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 0 & 6 & -12 & 3b-6a \\ 0 & 0 & 0 & 2c-2a-(3b-6a) \end{array}\right]$$

We need $\boxed{4a - 3b + 2c = 0}$ $\qquad \underbrace{2c - 2a - (3b - 6a)}_{4a - 3b + 2c}$

(otherwise we get $0 \cdot x + 0 \cdot y + 0 \cdot z = \text{nonzero}$)

E9 We can calculate,

$$\text{rref}\left[\begin{array}{ccc|c|c} 1 & -1 & 3 & 1 & 7 \\ 2 & 0 & 2 & 2 & 7 \\ 1 & 2 & -3 & 1 & 7 \end{array}\right] = \left[\begin{array}{ccc|c|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad (*)$$

these both follow from *

$$\text{rref}\left[\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 2 & 0 & 2 & 2 \\ 1 & 2 & -3 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x - y + 3z &= 1 \\ 2x + 2z &= 2 \\ x + 2y - 3z &= 1 \end{aligned}$$

Has solution $\boxed{\{(1-z, 2z, z) \mid z \in \mathbb{R}\}}$

$$\underbrace{\text{rref}\left[\begin{array}{ccc|c} 1 & -1 & 3 & 7 \\ 2 & 0 & 2 & 7 \\ 1 & 2 & -3 & 7 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]}_{\text{inconsistent system}}$$

$4a - 3b + 2c \neq 0$

$\boxed{\text{E10}}$ We're given that

$$\text{rref}\begin{bmatrix} 2 & 3 & 1 & 1 \\ 4 & 7 & 8 & 0 \\ 3 & 4 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 64/11 \\ 0 & 1 & 0 & -40/11 \\ 0 & 0 & 1 & 3/11 \end{bmatrix}$$

Let's interpret this reduction a few ways,

1.
$$\begin{aligned} 2x + 3y + z &= 1 \\ 4x + 7y + 8z &= 0 \\ 3x + 4y + 4z &= 4 \end{aligned} \implies \text{solution is } \left(\frac{64}{11}, \frac{-40}{11}, \frac{3}{11}\right)$$

2.
$$\begin{aligned} 2x + 3y &= 1 \\ 4x + 7y &= 8 \\ 3x + 4y &= 4 \end{aligned} \hookrightarrow \text{rref}\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 7 & 8 \\ 3 & 4 & 4 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$$\begin{aligned} 2x + 3y &= 1 \\ 4x + 7y &= 0 \\ 3x + 4y &= 4 \end{aligned} \hookrightarrow \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 7 & 0 \\ 3 & 4 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 64/11 \\ 0 & 1 & -40/11 \\ 0 & 0 & 3/11 \end{array}\right]$$

(both are inconsistent systems)

3.
$$\begin{aligned} 2x_1 + 3x_2 + x_3 + x_4 &= 0 \\ 4x_1 + 7x_2 + 8x_3 &= 0 \\ 3x_1 + 4x_2 + 4x_3 + 4x_4 &= 0 \end{aligned} \hookrightarrow \underbrace{\text{rref}\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 2 & 3 & 1 & 1 & 0 \\ 4 & 7 & 8 & 0 & 0 \\ 3 & 4 & 4 & 4 & 0 \end{array}\right] \end{array} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 64/11 & 0 \\ 0 & 1 & 0 & -40/11 & 0 \\ 0 & 0 & 1 & 3/11 & 0 \end{array}\right]}_{\text{we can adjoin column of zeros, it doesn't change the rest of it.}}$$

$$\begin{aligned} x_1 &= (-64/11)\,x_4 \\ x_2 &= (40/11)\,x_4 \\ x_3 &= (-3/11)\,x_4 \end{aligned}$$

for any $x_4 \in \mathbb{R}$

continued ↘

E10 continued

$$x_1 = \frac{-64}{11} x_4$$

$$x_2 = \frac{40}{11} x_4$$

$$x_3 = \frac{-3}{11} x_4$$

$$\text{Solution Set} = \left\{ \left(\frac{-64}{11} x_4, \frac{40}{11} x_4, \frac{-3}{11} x_4, x_4 \right) \middle| x_4 \in \mathbb{R} \right\}$$

$$= \left\{ \frac{x_4}{11}(-64, 40, -3, 11) \middle| x_4 \in \mathbb{R} \right\}$$

$$= \left\{ t(-64, 40, -3, 11) \middle| t \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ (-64, 40, -3, 11) \right\}$$

Def^n/ Given vectors $v_1, v_2, \ldots, v_m$ we say $\sum_{i=1}^{m} c_i v_i = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$ is a linear combination of $v_1, \ldots, v_m$.

$$\underbrace{\text{span}\{v_1, v_2, \ldots, v_m\}}_{\text{set of all possible linear combinations of } \{v_1, \ldots, v_m\}} = \left\{ \sum_{i=1}^{m} c_i v_i \middle| c_1, \ldots, c_m \in \mathbb{R} \right\}$$

$\boxed{\text{E11}}$ Write the solution set of

$$x_1 + 3x_2 - 4x_3 + x_4 = 0$$
$$x_2 + 3x_2 + 6x_3 + 4x_4 = 0$$

in terms of the <u>span</u> of appropriate vectors.

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 3 & -4 & 1 & 0 \\ 1 & 3 & 6 & 4 & 0 \end{array}\right] \end{array} \xrightarrow{r_2 - r_1} \left[\begin{array}{cccc|c} 1 & 3 & -4 & 1 & 0 \\ 0 & 0 & 10 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{r_1 + \frac{4}{10} r_2} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 11/5 & 0 \\ 0 & 0 & 10 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{r_2/10} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 11/5 & 0 \\ 0 & 0 & 1 & 3/10 & 0 \end{array}\right]$$

$$x_1 + 3x_2 + \frac{11}{5} x_4 = 0$$
$$x_3 + \frac{3}{10} x_4 = 0$$

pivotal variables $\begin{cases} x_1 = -3x_2 - \frac{11}{5} x_4 \\ x_3 = -\frac{3}{10} x_4 \end{cases}$

x_2, x_4 are non-pivotal variables, they're <u>free</u>

$$\text{Sol}^{n}\text{ Set} = \left\{ \left(-3x_2 - \tfrac{11}{5} x_4,\ x_2,\ -\tfrac{3}{10} x_4,\ x_4\right) \,\middle|\, x_2, x_4 \in \mathbb{R} \right\}$$
$$= \left\{ x_2(-3,1,0,0) + x_4\left(-\tfrac{11}{5}, 0, -\tfrac{3}{10}, 1\right) \,\middle|\, x_2, x_4 \in \mathbb{R} \right\}$$
$$= \boxed{\text{span}\left\{ (-3,1,0,0), \left(-11/5, 0, -3/10, 1\right) \right\}}$$