LECTURE 30 : REAL SPECTRAL THEOREM

This theorem is easy to remember and it gives us great insight,

Th$^{\underline{m}}$ (REAL SPECTRAL)

Let $A \in \mathbb{R}^{n\times n}$, $A^T = A \Leftrightarrow A$ has orthonormal eigenbasis.

Proof: $(\Rightarrow)$ Left to another course.

$(\Leftarrow)$ Suppose $\beta = \{v_1, v_2, \dots, v_n\}$ is orthonormal and $Av_j = \lambda_j v_j$ for some $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ (possibly repeated) then $[\beta]^{-1} = [\beta]^T$ since $([\beta]^T[\beta])_{ij} = v_i \cdot v_j = \delta_{ij}$ and as we've shown in an earlier Lecture,

$[\beta]^T A [\beta] = D = \text{Diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$. Therefore,

$[\beta][\beta]^T A [\beta][\beta]^T = [\beta] D [\beta]^T \Rightarrow A = [\beta] D [\beta]^T$

and we find $A^T = ([\beta]^T)^T D^T [\beta]^T = [\beta] D [\beta]^T = A.$ //

Let's look at an abstract 3x3 example, I'll introduce projection matrices which we can build from the orthonormal eigenbasis.

E1 Suppose $A \in \mathbb{R}^{3\times 3}$ with orthonormal eigenbasis v_1, v_2, v_3 with e-values $\lambda_1, \lambda_2, \lambda_3$ respectively,

$$Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_2 v_2, \quad Av_3 = \lambda_3 v_3$$

Then construct,

$$E_1 = v_1 v_1^T \quad \Rightarrow \quad E_1 v_1 = v_1 v_1^T v_1 = v_1 \qquad \because v_1 \cdot v_1 = 1$$

$$E_2 = v_2 v_2^T \qquad E_1 v_2 = v_1 v_1^T v_2 = 0 \qquad \left.\begin{matrix} v_1 \cdot v_2 = 0 \\ v_1 \cdot v_3 = 0 \end{matrix}\right\}$$

$$E_3 = v_3 v_3^T \qquad E_1 v_3 = v_1 v_1^T v_3 = 0$$

We can argue:

$$\boxed{A = \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3}$$

Matrices like E_1, E_2, E_3 have special properties which follow from $V_i \cdot V_j = V_i^T V_j = \delta_{ij}$

Lemma: given orthonormal vectors $V_1, V_2, \dots, V_n$ if we construct $E_i = V_i V_i^T$ for $i = 1, 2, \dots, n$ then these matrices have

1.) $E_i E_j = \delta_{ij} E_i$

2.) $E_1 + E_2 + \cdots + E_n = I_n$

Proof: the key is to use $V_i^T V_j = \delta_{ij}$. Consider, for (1.)

$$\begin{aligned} E_i E_j &= (V_i V_i^T)(V_j V_j^T) \\ &= V_i \, \delta_{ij} V_j^T \\ &= \delta_{ij} V_i V_j^T = \begin{cases} 0 & \text{if } i \neq j \\ V_i V_i^T & \text{if } i = j \end{cases} \\ &= \underline{\delta_{ij} E_i}. \end{aligned}$$

To prove (2.) we note that $A = B$ iff $AV_i = BV_i$ for $i = 1, 2, \dots, n$ since $V_1, \dots, V_n$ is basis for $\mathbb{R}^n$.

Consider then,

$$\begin{aligned} (E_1 + E_2 + \cdots + E_n) V_j &= \sum_{i=1}^n E_i V_j \\ &= \sum_{i=1}^n V_i V_i^T V_j \\ &= \sum_{i=1}^n V_i \delta_{ij} \\ &= V_j = I_n V_j \quad \forall j = 1, 2, \dots, n \end{aligned}$$

Thus $E_1 + E_2 + \cdots + E_n = I_n$ as claimed. //

We also should notice $E_i v_j = v_i v_i^T v_j = \delta_{ij} v_i$
Thus we find the eigenvalues of E_i are 0 and 1,

Proposition: $\text{Null}(E_i) = \text{span}\{v_j \mid j \neq i\} = \mathcal{E}_{\lambda=0}$ whereas $\text{Null}(E_i - I) = \text{span}\{v_i\}$ thus $\dim(\text{Null}(E_i)) = n-1$ and $\dim(\text{Null}(E_i - I)) = 1$

Therefore, we can use $E_1, E_2, \dots, E_n$ as above to build a matrix A with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$

Thm: with $E_i = v_i v_i^T$ for $v_1, v_2, \dots, v_n$ orthonormal if $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ then

$$A = \lambda_1 E_1 + \lambda_2 E_2 + \cdots + \lambda_n E_n$$

has eigenvectors $v_1, v_2, \dots, v_n$ with e-values $\lambda_1, \lambda_2, \dots, \lambda_n$.

E2 following E1, build (3×3) matrix with eigenvalues 6, 7 and -2 by

$$A = 6E_1 + 7E_2 - 2E_3$$

However, this is not unique,

$$B = 7E_1 - 2E_2 + 6E_3$$

there are many ways to build a matrix with eigenvalues 6, 7 and -2. For given orthonormal basis v_1, v_2, v_3 we can build $E_1 = v_1 v_1^T$, $E_2 = v_2 v_2^T$, $E_3 = v_3 v_3^T$

Remark: matrices built as in **E2** will be symmetric.

Let's examine how the Lemma on p. 2 works out for the standard basis $e_1, e_2, e_3, \ldots, e_n$

$$E_1 = e_1 e_1^T = e_1 [1, 0, \ldots, 0] = [e_1 | 0 | \cdots | 0]$$

$$E_2 = e_2 e_2^T = e_2 [0, 1, \ldots, 0] = [0 | e_2 | \cdots | 0]$$

$$\vdots$$

$$E_n = e_n e_n^T = e_n [0, 0, \ldots, 1] = [0 | \cdots | 0 | e_n]$$

Of course $E_1 + E_2 + \cdots + E_n = [e_1 / e_2 / \cdots / e_n] = I_n$. Makes sense.

$\boxed{E3}$ The matrix $A = \begin{bmatrix} 0 & 0 & -4 \\ 2 & 4 & 2 \\ 2 & 0 & 6 \end{bmatrix}$ has eigenvectors

$V_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $V_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $V_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ with

eigenvalues $\lambda_1 = 4$, $\lambda_2 = 4$ and $\lambda_3 = 2$. We have $\mathcal{E}_{\lambda_1 = 4} = \text{span}\{V_1, V_2\}$ and $\mathcal{E}_{\lambda_3 = 2} = \text{span}\{V_3\}$.

- Can we obtain an orthonormal basis for $\mathcal{E}_{\lambda_1 = 4}$?

Well, yes, $u_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ gives $\mathcal{E}_{\lambda_1 = 4} = \text{span}\{u_1, u_2\}$

- Can we obtain an orthonormal basis for $\mathcal{E}_{\lambda_3 = 2} = \text{span}\{(-2,1,1)\}$

Again, yes, $\mathcal{E}_{\lambda_3 = 2} = \text{span}\left\{\frac{1}{\sqrt{6}}(-2,1,1)\right\}$.

- Can we join our orthonormal bases for $\mathcal{E}_{\lambda_1}$ & $\mathcal{E}_{\lambda_3}$ to form orthonormal basis for $\mathcal{E}_{\lambda_1} + \mathcal{E}_{\lambda_3} = \mathbb{R}^3$

No. $A^T \neq A$ so $\nexists$ an orthonormal eigenbasis for A.

$$\mathcal{E}_{\lambda_1} \cap \mathcal{E}_{\lambda_3} = \{0\} \text{ so } \mathcal{E}_{\lambda_1} \oplus \mathcal{E}_{\lambda_2} = \mathbb{R}^3$$

However, $(\mathcal{E}_{\lambda_1})^\perp \neq \mathcal{E}_{\lambda_3}$

$\text{Th}^{\underline{m}}$/ Given $A^T = A$ and suppose $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues with multiplicities $n_1, n_2, \ldots, n_k$ then if $\mathcal{E}_j = \text{Null}(A - \lambda_j I)$ then $\mathcal{E}_1 + \mathcal{E}_2 + \cdots + \mathcal{E}_k = \mathbb{R}^n$ and $(\mathcal{E}_j)^\perp = \mathcal{E}_1 + \cdots + \mathcal{E}_{j-1} + \mathcal{E}_{j+1} + \cdots + \mathcal{E}_k$ hence

$$\mathcal{E}_j \cap (\mathcal{E}_1 + \cdots + \mathcal{E}_{j-1} + \mathcal{E}_{j+1} + \cdots + \mathcal{E}_k) = \{0\} \text{ and}$$

$$\mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_k = \mathbb{R}^n$$ (algebraic = geometric here)

In other words, the eigenspaces of a symmetric matrix form orthogonal complements

$\boxed{\text{E4}}$ $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ has $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ with $\lambda_1 = 3$

whereas $\text{Null}(A) = \text{span}\{(1,-1,0), (1,0,-1)\}$ shows the geometric multiplicity of $\lambda_2 = 0$ is $\nu_2 = 2$.

$$\mathcal{E}_2 = \text{span}\{\underbrace{(1,-1,0), (1,0,-1)}_{\text{not an orthonormal basis for } \mathcal{E}_2}\}$$

Consider,

$$\text{Orth}_{(1,-1,0)}(1,0,-1) = (1,0,-1) - \frac{(1,-1,0)\cdot(1,0,-1)}{(1,-1,0)\cdot(1,-1,0)}(1,-1,0) = (1,0,-1) - \tfrac{1}{2}(1,-1,0)$$
$$= (1 - 1/2, 1/2, -1)$$
$$= \tfrac{1}{2}(1,1,-2)$$

Hence

$\left\{ \frac{1}{\sqrt{2}}(1,-1,0), \frac{1}{\sqrt{6}}(1,1,-2) \right\} \leftarrow$ orthonormal basis for $\mathcal{E}_2$

Then $\beta = \left\{ \underbrace{\frac{1}{\sqrt{3}}(1,1,1)}_{v_1}, \underbrace{\frac{1}{\sqrt{2}}(1,-1,0)}_{v_2}, \underbrace{\frac{1}{\sqrt{6}}(1,1,-2)}_{v_3} \right\}$ orthonormal eigenbasis for A. (Here $A = 3v_1v_1^T + 0v_2v_2^T + 0v_3v_3^T$)

Next Lecture we apply the Real Spectral Th$^{\underline{m}}$ to study quadratic forms. Let's examine one example here to get started.

$\boxed{E5}$ Consider the formula

$$Q(x,y,z) = 4x^2 + 3y^2 + 4z^2 + 4xy + 2xz + 4yz$$

We can write such formula as the product of $[x,y,z]$ row vector, a <u>symmetric</u> <u>matrix</u> and the column vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$,

$$Q(x,y,z) = [x,y,z]\underbrace{\begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

we write $Q(v) = v^TAv$. Some calculation reveals A has orthonormal eigenbasis

$$\bar{\beta} = \left\{ \tfrac{1}{\sqrt{6}}(1,-2,1), \tfrac{1}{\sqrt{2}}(1,0,-1), \tfrac{1}{\sqrt{3}}(1,1,1) \right\}$$

with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 7$. Recall, coordinate map $(\bar{x}, \bar{y}, \bar{z}) = \Phi_{\bar{\beta}}(x,y,z) = [\bar{\beta}]^{-1}(x,y,z)$ and since $[\bar{\beta}]^{-1} = [\bar{\beta}]^T$ as $[\bar{\beta}]^T[\bar{\beta}] = I$ for β orthonormal.
(we call such coordinates $\bar{x}, \bar{y}, \bar{z}$ eigencoordinates)

$$\begin{aligned} Q(v) = v^TAv &= v^T[\bar{\beta}][\bar{\beta}]^TA[\bar{\beta}][\bar{\beta}]^Tv \\ &= ([\bar{\beta}]^Tv)([\bar{\beta}]^TA[\bar{\beta}])([\bar{\beta}]^Tv) \\ &= [\bar{x},\bar{y},\bar{z}]\begin{bmatrix} 1 & & \\ & 3 & \\ & & 7 \end{bmatrix}\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = \bar{x}^2 + 3\bar{y}^2 + 7\bar{z}^2 \end{aligned}$$

<u>Remark:</u> $\bar{x}^2 + 3\bar{y}^2 + 7\bar{z}^2 = 1$ is clearly an ellipsoid, $4x^2 + 3y^2 + 4z^2 + 4xy + 2xy + 4yz = 1$ is not clearly anything. Changing to eigencoordinates uncouples terms, it removes cross-terms.