LECTURE 32: SINGULAR VALUE DECOMPOSITION

Given any real $m \times n$ matrix we can find a SVD which shows $v \mapsto Av$ can be understood as a 3-step process of rotation / dilation and/or projection / rotation. (There is also a SVD for complex matrices, but I'll keep it real in here)

$$\boxed{A = U \Sigma V^T}$$

Let's begin with $A \in \mathbb{R}^{m \times n}$, I'll describe the algorithm to calculate the SVD of A.

$$\underbrace{(A^TA)^T = A^T(A^T)^T = A^TA}$$

apparently A^TA is symmetric $(n \times n)$ matrix

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be eigenvalues of A^TA

with e-vectors $v_1, v_2, \ldots, v_n$ orthonormal for A^TA

Construct $V = [v_1 | v_2 | \cdots | v_n] \in O(n, \mathbb{R})$

It turns out $\lambda_j \geq 0$ so we can also construct singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ by setting $\sigma_1 = \sqrt{\lambda_1}, \sigma_2 = \sqrt{\lambda_2}, \ldots, \sigma_n = \sqrt{\lambda_n}$.

Construct $\Sigma \in \mathbb{R}^{m \times n}$ as follows

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_m & \end{bmatrix}$$

(there cannot be more than m nonzero σ_j since the $\operatorname{rank}(A^TA) \leq \operatorname{rank}(A) \leq m$)

Next, construct $(m \times m)$ matrix $U = [u_1 | u_2 | \cdots | u_m]$ by setting $u_j = \frac{1}{\sigma_j} A v_j$ for each $\sigma_j \neq 0$ then complete U by adjoining orthonormalized vectors as needed to form $U \in O(m, \mathbb{R})$.

$$A = U \Sigma V^T$$

$$U = [\underbrace{u_1 | u_2 | \cdots | u_m}_{\text{LEFT SINGULAR VECTORS}}]$$

where $u_j = \frac{1}{\sigma_j} A v_j$ for each nonzero singular value $\sigma_1, \sigma_2, \ldots, \sigma_r$

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_r & \\ & & & & \end{bmatrix} \in \mathbb{R}^{m \times n}$$

$$V = [\underbrace{v_1 | v_2 | \cdots | v_n}_{\text{RIGHT SINGULAR VECTORS}}] \in \mathbb{R}^{n \times n}$$

where $v_1, \ldots, v_n$ are eigenvectors of $A^T A$

$$A^T A v_j = \lambda_j v_j$$

where $\sigma_j = \sqrt{\lambda_j}$

$A^T A v_j = 0$ for $j = r+1, \ldots, n$

Consider $AA^T \in \mathbb{R}^{m \times m}$ is symmetric $(AA^T)^T = AA^T$

$$AA^T u_j = \frac{1}{\sigma_j} AA^T A v_j = \frac{1}{\sigma_j} A \lambda_j v_j = \lambda_j u_j$$

(we find $u_1, u_2, \ldots, u_r$ are e-vectors of AA^T with nonzero e-values $\lambda_1, \lambda_2, \ldots, \lambda_r$ which are also the nonzero e-values for $A^T A$)

$\boxed{E1}$ $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$$

$\tilde{V}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has $\lambda_1 = 15$ since $A\tilde{V}_1 = 15\tilde{V}_1$

$V_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $V_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$V_1 \times V_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = V_3$ $\underline{\lambda_2 = \lambda_3 = 0}$.

$$AA^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}$$

$$\det\begin{bmatrix} 3-\lambda & 6 \\ 6 & 12-\lambda \end{bmatrix} = (\lambda-3)(\lambda-12)-36 = \lambda^2 - 15\lambda = \lambda(\lambda-15)$$

$$AA^T - 15I = \begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix} \longrightarrow \tilde{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \longrightarrow u_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

we have two choices for u_2, I choose $u_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

<u>Remark</u>: I found u_1 differently than I previously proposed, let's see if $u_1 = \frac{1}{\sigma_1}AV_1 = \frac{1}{\sqrt{15}}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{45}}\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ which gives $u_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ just as we found.

$$\boxed{A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} \sqrt{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix}}_{V^T}}$$

$\boxed{E2}$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 4 \end{bmatrix} \text{ has } \underline{\lambda_1 = 6,\ \lambda_2 = 2,\ \lambda_3 = 0}$$

$$u_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\quad u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},\quad u_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\sigma_1 = \sqrt{6},\quad \sigma_2 = \sqrt{2}$$

<u>Remark</u>: if $V_j = \frac{1}{\sigma_j} A^T u_j$ for $AA^T u_j = \lambda_j u_j$ where $\lambda_j \neq 0$ then observe $A^T A V_j = \frac{1}{\sigma_j} A^T A A^T u_j = \lambda_j \left(\frac{1}{\sigma_j} A^T u_j \right) = \lambda_j V_j$ so we can generate right singular vectors from left.

$$V_1 = \frac{1}{(\sqrt{6})^2}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$V_2 = \frac{1}{(\sqrt{2})^2}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

Then we need to extend orthonormally, $V_3 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$, $V_4 = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$$\boxed{A = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix}\begin{bmatrix} \sqrt{6} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}$$

$\boxed{E3}$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

$\lambda_1 = 4, \quad v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\lambda_2 = 2, \quad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\sigma_1 = 2 \qquad \sigma_2 = \sqrt{2}$

$$u_1 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\underbrace{u_3 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}}_{\text{makes } u_1, u_2, u_3 \text{ orthonormal}}$$

$$\boxed{A = \underbrace{\begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{V^T}}$$