LECTURE 3: LINEAR SYSTEMS VIA MATRIX COLUMN PRODUCTS AND LINEAR COMBINATIONS & DEPENDENCE

Defn/ $\text{span}\{V_1, V_2, \dots, V_n\} = \{\underbrace{c_1V_1 + c_2V_2 + \cdots + c_nV_n}_{\text{this is a linear combination of } V_1, V_2, \dots, V_n} \mid c_1, c_2, \dots, c_n \in \mathbb{R}\}$

this is a <u>linear</u> <u>combination</u> of $V_1, V_2, \dots, V_n$

When it is not possible to write one vector in a given set as a linear combination of the remaining vectors then that set is called <u>linearly</u> <u>independent</u>. If it is possible to write one vector as linear comb. of other vectors in a given set then that set is said to be linearly dependent. To be more careful than these words, here is <u>the</u> definition,

Defn/ $S = \{V_1, V_2, \dots, V_n\}$ is <u>LINEARLY</u> <u>INDEPENDENT</u> OR <u>LI</u> if $c_1V_1 + c_2V_2 + \cdots + c_nV_n = 0 \Rightarrow c_1 = 0, \dots, c_n = 0.$ If S IS NOT LI then S IS <u>LINEARLY</u> <u>DEPENDENT</u>

Thm/ If $S = \{V_1, \dots, V_n\}$ IS NOT LI then there exists $V_j \in S$ for which $V_j = \sum_{i \neq j} c_i V_i$. That is, $V_j \in \text{span}\underbrace{\{V_1, V_2, \dots, V_{j-1}, V_{j+1}, \dots, V_n\}}_{S - \{V_j\}}$

$V_j \in \text{span}(S - \{V_j\})$

Proof: Suppose $S = \{V_1, V_2, .., V_n\}$ is not LI. Then there exist constants $c_1, c_2, .., c_n$ (not all zero) for which

$$c_1 V_1 + c_2 V_2 + \cdots + c_n V_n = 0$$

Suppose $c_j \neq 0$ then $c_j V_j = -c_1 V_1 - \cdots - c_{j-1} V_{j-1} - c_{j+1} V_{j+1} - \cdots c_n V_n$

Thus $V_j = \frac{-c_1}{c_j} V_1 - \cdots - \frac{c_{j-1}}{c_j} V_{j-1} - \frac{c_{j+1}}{c_j} V_{j+1} - \cdots - \frac{c_n}{c_j} V_n$

Hence $V_j \in \text{span}\{V_1, .., V_{j-1}, V_{j+1}, .., V_n\}$. //

$\boxed{E1}$ $S = \{x+1, x-1, x\}$ is <u>linearly dependent</u> since $\frac{1}{2}(x+1) + \frac{1}{2}(x-1) = x$.

$\boxed{E2}$ $S = \{1, x, x^2\}$ is LI

Suppose $\underline{c_1(1) + c_2 x + c_3 x^2 = 0}$ ⓢ

Feed ⓢ $x=0$ to get $\underline{c_1 = 0}$. Note $\frac{dⓢ}{dx}$ yields $c_2 + 2c_3 x = 0$ and set $x = 0$ to get $\underline{c_2 = 0}$. Then plug $x = 1$ into ⓢ to find $c_3(1)^2 = 0 \therefore \underline{c_3 = 0}$. Consequently $c_1 = c_2 = c_3 = 0$ hence S is LI.

$\boxed{E3}$ Let $S = \{f(x) \in P_3(\mathbb{R}) \mid f'(1) = 0\}$

$f(x) \in P_3(\mathbb{R})$ means $f(x) \in \text{span}\{1, x, x^2, x^3\}$ which means $f(x) = a + bx + cx^2 + dx^3$. Then $f(x) \in S$ requires $f'(x) = b + 2cx + 3dx^2$ to evaluate at $x = 1$ to give $f'(1) = b + 2c + 3d = 0$. Then $b = -2c - 3d \Rightarrow \underline{f(x) = a + (-2c-3d)x + cx^2 + dx^3}$

$$f(x) = a(1) + c(-2x + x^2) + d(-3x + x^3)$$

E3 continued

We found $f(x) \in S$ implies that

$$f(x) = \underbrace{a(1) + c(-2x + x^2) + d(x^3 - 3x)}$$

$$\therefore \; f(x) \in \text{span}\{1, x^2 - 2x, x^3 - 3x\}.$$

Let us return once more to systems of equations, we introduce further matrix notations, this is probably overdue at this point

Def^n/ An $m \times n$ matrix $A \in \mathbb{R}^{m \times n}$ has the form

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} = [\text{col}_1(A) | \text{col}_2(A) | \cdots | \text{col}_n(A)]$$

$$= \begin{bmatrix} \text{row}_1(A) \\ \hline \text{row}_2(A) \\ \hline \vdots \\ \hline \text{row}_m(A) \end{bmatrix}$$

Here $(\text{row}_i(A))_j = A_{ij}$ whereas $(\text{col}_j(A))_i = A_{ij}$. and $1 \le i \le m$ whereas $1 \le j \le n$. If $m = n$ then A is called <u>square</u>

Special case of matrix is a <u>column</u> <u>vector</u>

Def^n/ $v = (v_1, v_2, \dots, v_n) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^{n \times 1} = \mathbb{R}^n$

<u>Remark:</u> $[w_1, w_2, \dots, w_m] \in \mathbb{R}^{1 \times m}$ and $[w_1, \dots, w_m] \neq (w_1, \dots, w_m)$.

We can multiply an $(m\times n)$ matrix and an $(n\times 1)$ column vector to create an $(m\times 1)$ column vector

Def^{n}/
$$\underbrace{\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}}_{m\times n}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n \\ \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n \end{bmatrix}}_{m\times 1}$$

We can understand the above in terms of linear combo's,

Th^{m}/ $Ax = x_1\,\mathrm{col}_1(A) + x_2\,\mathrm{col}_2(A) + \cdots + x_n\,\mathrm{col}_n(A)$

Or, in terms of dot-products,

Th^{m}/ $$Ax = \begin{bmatrix} \mathrm{row}_1(A)\cdot x \\ \mathrm{row}_2(A)\cdot x \\ \vdots \\ \mathrm{row}_m(A)\cdot x \end{bmatrix}$$

Def^{n}/ $\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + \cdots + A_nB_n$ (dot-product for n-vectors)

Notice,

$$Ax = b \iff x_1\,\mathrm{col}_1(A) + x_2\,\mathrm{col}_2(A) + \cdots + x_n\,\mathrm{col}_n(A) = b$$
$$\iff b \in \mathrm{span}\{\mathrm{col}_1(A), \ldots, \mathrm{col}_n(A)\}$$

Th^{m}/ The system of equations $Ax = b$ has a solution if and only if $b \in \mathrm{span}\{\mathrm{col}_1(A), \ldots, \mathrm{col}_n(A)\}$

E4

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\cdot\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 0\cdot\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + 0\cdot\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \langle 1,2,3\rangle \cdot \langle 1,0,0\rangle \\ \langle 4,5,6\rangle \cdot \langle 1,0,0\rangle \\ \langle 7,8,9\rangle \cdot \langle 1,0,0\rangle \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

$$\left.\begin{aligned} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \end{aligned}\right\}$$

Thm/ $Ae_i = col_i(A)$ for any $m\times n$ matrix A

E5

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = x\begin{bmatrix} 1 \\ 5 \end{bmatrix} + y\begin{bmatrix} 2 \\ 6 \end{bmatrix} + z\begin{bmatrix} 3 \\ 7 \end{bmatrix} + w\begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} x \\ 5x \end{bmatrix} + \begin{bmatrix} 2y \\ 6y \end{bmatrix} + \begin{bmatrix} 3z \\ 7z \end{bmatrix} + \begin{bmatrix} 4w \\ 8w \end{bmatrix}$$

$$= \begin{bmatrix} x + 2y + 3z + 4w \\ 5x + 6y + 7z + 8w \end{bmatrix}$$

$\boxed{E6}$ Is $(1,2,2,3) \in \operatorname{span}\left\{ \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}3\\0\\3\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\2\end{bmatrix} \right\}$?

$$\underbrace{x\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + y\begin{bmatrix}3\\0\\3\\0\end{bmatrix} + z\begin{bmatrix}0\\1\\1\\2\end{bmatrix}}_{\begin{bmatrix}1&3&0\\1&0&1\\1&3&1\\1&0&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}} = \begin{bmatrix}1\\2\\2\\3\end{bmatrix}$$

can we solve this?

$$\begin{bmatrix}1&3&0\\1&0&1\\1&3&1\\1&0&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\2\\2\\3\end{bmatrix}$$

$$\operatorname{rref}\left[\begin{array}{ccc|c}1&3&0&1\\1&0&1&2\\1&3&1&2\\1&0&2&3\end{array}\right] = \left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&0\\0&0&1&1\\0&0&0&0\end{array}\right] \begin{array}{l}\leftrightarrow x=1\\ \leftrightarrow y=0\\ \leftrightarrow z=1\\ \\ \end{array}$$

$$\therefore \begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \begin{bmatrix}0\\1\\1\\2\end{bmatrix} = \begin{bmatrix}1\\2\\2\\3\end{bmatrix}$$

this shows that $(1,2,2,3)$ is in $\operatorname{span}\left\{ \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}3\\0\\3\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\2\end{bmatrix} \right\}$

Remark: we should look at my typed lecture notes, see Example 2.4.10 on pg. 40 Nice chemistry example