LECTURE 4: INDEX NOTATION & MATRIX ALGEBRA

Two matrices can only be equal if they have the same size and matching entries,

Def^{n}/ We say $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$ are equal and write $A = B$ iff $A_{ij} = B_{ij}$ $\underbrace{\text{for all } i,j}_{\substack{1\le i\le m \\ 1\le j\le n}}$

Addition, subtraction and scalar multiplication are defined component-wise,

Def^{n}/ Let $A, B \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}$ then we define $A \pm B,\ cA \in \mathbb{R}^{m\times n}$ by

$$(A \pm B)_{ij} = A_{ij} \pm B_{ij}$$

$$(cA)_{ij} = cA_{ij}$$

for all $1 \le i \le m$ and $1 \le j \le n$

$\boxed{\text{E1}}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 5 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 6 & 7 \end{bmatrix}$$

$$3\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 & 12 \\ 15 & 18 & 21 & 24 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x & 2 \\ 3 & y \end{bmatrix} \Rightarrow \left(\begin{matrix} a = x, & b = 2 \\ c = 3, & d = y \end{matrix} \right)$$

Def^n/ Let $A \in \mathbb{R}^{m\times n}$ then $A^T \in \mathbb{R}^{n\times m}$ is <u>transpose</u> of A defined by $(A^T)_{ji} = A_{ij}$ $1 \le i \le m$, $1 \le j \le n$.

E2

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$e_1^T = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T = [1, 0, \dots, 0]$$

$$\begin{bmatrix} 3 & 4 \\ 4 & 2 \end{bmatrix}^T = \begin{bmatrix} 3 & 4 \\ 4 & 2 \end{bmatrix} \quad \text{(symmetric matrix)}$$

PROPERTIES OF TRANSPOSE

$$\boxed{(A^T)^T = A, \ (A+B)^T = A^T + B^T, \ (A-B)^T = A^T - B^T, \ (cA)^T = cA^T}$$

<u>Proof</u>: $((A^T)^T)_{ij} = (A^T)_{ji} = A_{ij}$ $\therefore$ $\underline{(A^T)^T = A}$.

$$((A+B)^T)_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = (A^T)_{ij} + (B^T)_{ij}$$
$$= (A^T + B^T)_{ij}. \quad *$$

Hence $(A+B)^T = A^T + B^T$ since $*$ shows these matrices have same components.

Proofs of $(A-B)^T = A^T - B^T$ and $(cA)^T = cA^T$ are easily accomplished by similar arguments. //

Def^n/ $A \in \mathbb{R}^{n\times n}$ is <u>symmetric</u> if $A^T = A$ and is <u>antisymmetric</u> or <u>skew-symmetric</u> if $A^T = -A$

E3

$$\begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}^T\right) + \frac{1}{2}\left(\begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}^T\right)$$

$$= \frac{1}{2}\begin{bmatrix} 8 & 9 \\ 9 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 4 & 9/2 \\ 9/2 & 2 \end{bmatrix}}_{\text{symmetric}} + \underbrace{\begin{bmatrix} 0 & 3/2 \\ -3/2 & 0 \end{bmatrix}}_{\text{antisymmetric}}$$

Remark: given $A \in \mathbb{R}^{n \times n}$ we can always play the game just shown in E3, $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$.

Notice $\left(\frac{1}{2}(A+A^T)\right)^T = \frac{1}{2}(A^T + A^{TT}) = \frac{1}{2}(A+A^T)$ is symmetric

whereas $\left(\frac{1}{2}(A-A^T)\right)^T = \frac{1}{2}(A^T - A^{TT}) = -\frac{1}{2}(A-A^T)$ is antisymm.

Def: $A \in \mathbb{R}^{m \times n}$ is <u>upper triangular</u> if all its entries below the diagonal are zero; $A_{ij} = 0$ for $i > j$. Likewise, A is <u>lower triangular</u> if $A_{ij} = 0$ for $i < j$.

$$\underbrace{A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ & A_{22} & \cdots & A_{2n} \\ & 0 & \ddots & \vdots \\ & & & A_{nn} \end{bmatrix}}_{\text{upper-}\Delta} \qquad \underbrace{A = \begin{bmatrix} A_{11} & & & 0 \\ A_{21} & A_{22} & & \\ \vdots & & \ddots & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}}_{\text{lower-}\Delta}$$

if A is both up-Δ and lower-Δ then A is <u>diagonal</u> meaning $A_{ij} = 0$ for all $i \neq j$.

I'm including Δ-matrices here so we can use them as examples.

MATRIX MULTIPLICATION

Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ then we define their product $AB \in \mathbb{R}^{m\times p}$ as follows,

$$\boxed{(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = row_i(A)\cdot col_j(B)}$$

With the understanding that if $n=1$ we don't need the sum and if $m=1$ then we don't need the index i and if $p=1$ then we don't need the index j.

E4

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{bmatrix}$$

$(2\times 2)\,(2\times 1) : (2\times 1)$

$$\boxed{(Ax)_i = \sum_{k=1}^{n} A_{ik}x_k}$$

$(m\times n)(n\times 1) : (m\times 1)$

E5

$$[y_1, y_2, y_3]\begin{bmatrix} 4 & 7 \\ 5 & 8 \\ 6 & 9 \end{bmatrix} = [4y_1 + 5y_2 + 6y_3,\ 7y_1 + 8y_2 + 9y_3]$$

$(1\times 3)\,(3\times 2) : (1\times 2)$

$$\boxed{(yA)_j = \sum_{k=1}^{m} y_k A_{kj}}$$

$(1\times m)(m\times n) : (1\times n)$

E6

$$[1, 2, 3]\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1\cdot 4 + 2\cdot 5 + 3\cdot 6 = 32$$

$(1\times 3)(3\times 1) : (1\times 1)$

$$\boxed{yx = \sum_{k=1}^{n} y_k x_k \quad y \in \mathbb{R}^{1\times n} \text{ and } x \in \mathbb{R}^{n\times 1}}$$

Th^m/ Given $v, w \in \mathbb{R}^n$ then $v^T w = v\cdot w = \sum_{k=1}^{n} v_k w_k$

I'm working to exhaust unusual cases, we'll soon turn to properties of matrix multiplication.

For $v \in \mathbb{R}^m$, $w \in \mathbb{R}^{1\times n}$

$$(vw)_{ij} = v_i w_j$$

for $1 \le i \le m$ and $1 \le j \le n$

$\boxed{E7}$ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}[4, 0, 6, 8] = \begin{bmatrix} 4 & 0 & 6 & 8 \\ 8 & 0 & 12 & 16 \\ 12 & 0 & 18 & 24 \end{bmatrix}$

$(3\times 1)(1\times 4) : (3\times 4)$

Very well, let's turn to less odd examples, notice the product of lower-Δ matrices is lower-Δ.

$\boxed{E8}$ $$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{B} = \begin{bmatrix} 2 & 3 & 10 \\ 0 & 1 & 12 \\ 0 & 0 & 3 \end{bmatrix}$$

$(AB)_{ij} = \text{row}_i(A) \cdot \text{col}_j(B)$

$\text{row}_2(A) \cdot \text{col}_3(B) = \langle 0, 1, 4 \rangle \cdot \langle 1, 0, 3 \rangle = 0 + 0 + 4 \cdot 3 = 12.$

We can process $\boxed{E8}$ through several matrix column products

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = A\,\text{col}_1(B)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = A\,\text{col}_2(B)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ 3 \end{bmatrix} = A\,\text{col}_3(B)$$

We see $AB = A\,[\text{col}_1(B) \,|\, \text{col}_2(B) \,|\, \text{col}_3(B)]$

$$= \underbrace{[A\,\text{col}_1(B) \,|\, A\,\text{col}_2(B) \,|\, A\,\text{col}_3(B)]}_{\text{this illustrates the column-by-column multiplication rule}}$$

this illustrates the column-by-column multiplication rule

$$\boxed{\begin{array}{l}\text{Th}^{m}/\ \text{Let } A \in \mathbb{R}^{m\times n} \text{ and } B = [\bar{B}_1 | \bar{B}_2 | \cdots | \bar{B}_k] \in \mathbb{R}^{n\times k} \\ \text{then } AB = [A\bar{B}_1 | A\bar{B}_2 | \cdots | A\bar{B}_k]\end{array}}$$

<u>Proof:</u> $(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{n} A_{ik}(\bar{B}_j)_k = (A\bar{B}_j)_i$ //

Alternatively, let me try again with column notation,

$$(\text{col}_j(AB))_i = (AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{n} A_{ik}(\text{col}_j(B))_k$$

$$= (A\,\text{col}_j(B))_i$$

$$\therefore \boxed{\text{col}_j(AB) = A\,\text{col}_j(B)}$$

Recall we introduced the standard basis $e_1, e_2, \dots, e_n \in \mathbb{R}^n$ and we saw that $Ae_1 = \text{col}_1(A)$ and $Ae_2 = \text{col}_2(A)$ etc. thus $Ae_j = \text{col}_j(A)$ for each column of A

$$\boxed{\begin{array}{l}\text{Th}^{m}/\ \text{If } A \in \mathbb{R}^{m\times n} \text{ and } I = [e_1 | e_2 | \cdots | e_n] \in \mathbb{R}^{n\times n} \\ \text{then } AI = A\end{array}}$$

<u>Proof:</u>

$$\begin{aligned} AI &= A[e_1 | e_2 | \cdots | e_n] \\ &= [Ae_1 | Ae_2 | \cdots | Ae_n] \\ &= [\text{col}_1(A) | \text{col}_2(A) | \cdots | \text{col}_n(A)] \\ &= A. \end{aligned}$$

$$\boxed{\begin{array}{l}\text{Def}^{n}/\ I_n = [e_1 | e_2 | \cdots | e_n] \in \mathbb{R}^{n\times n} \text{ is} \\ \text{the } \underline{\text{identity matrix}};\ (I_n)_{ij} = \delta_{ij} \\ \text{for all } 1 \le i, j \le n.\end{array}}$$

Def/ If $A \in \mathbb{R}^{n\times n}$ has $B \in \mathbb{R}^{n\times n}$ such that $AB = I_n$ and $BA = I_n$ then A is <u>invertible</u> and we write $A^{-1} = B$

Remark: we do <u>not</u> write $\frac{1}{A}$ for A^{-1}.

Th/ If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad-bc \neq 0$ then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

<u>Proof</u>: Suppose $ad-bc \neq 0$ and calculate,

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\left[\begin{array}{c|c} da-bc & db-bd \\ \hline -ca+ac & -bc+ad \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\left[\begin{array}{c|c} ad-bc & -ab+ba \\ \hline cd-dc & -cb+da \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

thus $\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$. //

memorize this, it is the easiest solution for two equation two unknown problems!

E9 Solve $\begin{pmatrix} 3x - 8y = a \\ -x + 3y = b \end{pmatrix}$

$$Av = b$$
$$A^{-1}Av = A^{-1}b$$
$$Iv = A^{-1}b$$
$$\therefore\ v = A^{-1}b$$

$$\begin{bmatrix} 3 & -8 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} 3 & -8 \\ -1 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 3 & -8 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -8 \\ -1 & 3 \end{bmatrix}^{-1}\begin{bmatrix} a \\ b \end{bmatrix}$$

solution by multiplication by inverse.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{9-8}\begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a + 8b \\ a + 3b \end{bmatrix}$$

$$\therefore\ \boxed{(x, y) = (3a+8b,\ a+3b)}$$

We'll return to the problem of calculating A^{-1} for larger matrices in a future lecture. Let's wrap-up the basics of matrix arithmetic for today,

$\boxed{E10}$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{B} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{B}\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$$

$AB \neq BA$ these matrices do not commute.

$\boxed{E11}$

$$N_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad N_3^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N_3^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

(N_3 is said to be nilpotent of order 3) since $N_3, N_3^2 \neq 0$ yet $N_3^3 = 0$

Th$^{\underline{m}}$/ For matrices of appropriate size and constants c_1, c_2, c

1. $A + (B + C) = (A + B) + C$
2. $A(BC) = (AB)C$
3. $A + B = B + A$
4. $c(A+B) = cA + cB$
5. $(c_1 + c_2)A = c_1 A + c_2 A$
6. $c_1(c_2 A) = (c_1 c_2) A$
7. $1 \cdot A = A$
8. for $A \in \mathbb{R}^{m \times n}$, $I_m A = A = A I_n$
9. $A(B+C) = AB + AC$
10. $(A+B)C = AC + BC$

Remark: next time we'll look at a few proofs of these

Proof ③:
$$\begin{aligned} (A+B)_{ij} &= A_{ij} + B_{ij} &&: \text{def}^{\underline{n}} \text{ of } + \\ &= B_{ij} + A_{ij} &&: \text{prop. of } \mathbb{R} \\ &= (B+A)_{ij} &&: \text{def}^{\underline{n}} \text{ of } + \end{aligned}$$

$\therefore$ $A + B = B + A$ since the calculation above holds $\forall i,j$. //