Lecture 5: Standard Basis and Matrix Calculation

Index notation allows us to formulate simple proofs for matrices of arbitrary size.

Claim: for $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{p \times q}$ the product $A(BC) = (AB)C$ associativity of mat. mult.

$$
\begin{aligned}
(A(BC))_{ij} &= \sum_{k=1}^{n} A_{ik} (BC)_{kj} && : \text{def}^{\text{n}} \text{ of mat. mult.} \\
&= \sum_{k=1}^{n} A_{ik} \sum_{l=1}^{p} B_{kl} C_{lj} && : \text{def}^{\text{n}} \text{ of mat. mult.} \\
&= \sum_{k=1}^{n} \sum_{l=1}^{p} A_{ik} (B_{kl} C_{lj}) && : \text{property of } \textstyle\sum \\
&= \sum_{l=1}^{p} \sum_{k=1}^{n} (A_{ik} B_{kl}) C_{lj} && : \text{associativity of } \mathbb{R} \text{ and interchanging finite sums} \\
&= \sum_{l=1}^{p} (AB)_{il} C_{lj} && : \text{def}^{\text{n}} \text{ of matrix mult.} \\
&= ((AB)C)_{ij} && \text{for all } 1 \le i \le m,\ 1 \le j \le q
\end{aligned}
$$

$\therefore \ A(BC) = (AB)C.$ //

Claim: $A(B+C) = AB + AC$

$$
\begin{aligned}
(A(B+C))_{ij} &= \sum_{k} A_{ik} (B+C)_{kj} \\
&= \sum_{k} A_{ik} (B_{kj} + C_{kj}) \\
&= \sum_{k} A_{ik} B_{kj} + \sum_{k} A_{ik} C_{kj} \\
&= (AB)_{ij} + (AC)_{ij} \\
&= (AB + AC)_{ij}
\end{aligned}
$$

$\therefore \ \underline{A(B+C) = AB + AC}$ //

Let's look at two more index calculations

CLAIM: for $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ we have $(AB)^T = B^T A^T$

Proof:
$$
\begin{aligned}
((AB)^T)_{ij} &= (AB)_{ji} && : \text{def}^{\underline{n}} \text{ of transpose} \\
&= \sum_{k=1}^{n} A_{jk} B_{ki} && : \text{def}^{\underline{n}} \text{ of mat. mult.} \\
&= \sum_{k=1}^{n} (A^T)_{kj} (B^T)_{ik} && : \text{def}^{\underline{n}} \text{ of transpose} \\
&= \sum_{k=1}^{n} (B^T)_{ik} (A^T)_{kj} && : \text{real \#'s commute} \\
&= (B^T A^T)_{ij} && : \text{def}^{\underline{n}} \text{ of matrix mult.}
\end{aligned}
$$

Thus $(AB)^T = B^T A^T$ since the above holds $\forall i,j$. //

Def$^{\underline{n}}$/ The **TRACE** of a square matrix $A \in \mathbb{R}^{n\times n}$ is the sum of the diagonals; $\text{tr}(A) = \sum_{i=1}^{n} A_{ii}$

CLAIM: $\text{tr}(A + cB) = \text{tr}(A) + c\,\text{tr}(B)$

Proof: let $c \in \mathbb{R}$ and $A, B \in \mathbb{R}^{n\times n}$ then
$$
\begin{aligned}
\text{tr}(A + cB) &= \sum_{i=1}^{n} (A + cB)_{ii} && : \text{def}^{\underline{n}} \text{ of trace} \\
&= \sum_{i=1}^{n} (A_{ii} + cB_{ii}) && : \text{def}^{\underline{n}} \text{ of addition and scalar mult.} \\
&= \sum_{i=1}^{n} A_{ii} + c\sum_{i=1}^{n} B_{ii} && : \text{property of } \Sigma \\
&= \underline{\text{tr}(A) + c\,\text{tr}(B)} && : \text{def}^{\underline{n}} \text{ of trace.}
\end{aligned}
$$
//

Remark: you can show that $\text{tr}(AB) = \text{tr}(BA)$ for multipliable $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times m}$

Th^{m}/ Let $e_1, e_2, \dots, e_n$ be the standard basis for $\mathbb{R}^n$ and suppose $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$ then

$$Ae_i = \text{col}_i(A) \quad \text{and} \quad e_i^T B = \text{row}_i(B)$$

We've seen $Ae_i = \text{col}_i(A)$ in previous episodes. Take the transpose to obtain,

$$(Ae_i)^T = (\text{col}_i(A))^T$$

$$e_i^T A^T = \text{row}_i(A^T)$$

then set $B = A^T$ to obtain $\underline{e_i^T B = \text{row}_i(B)}$ .//

We can also apply the socks-shoes identity to the column-by-column rule $\boxed{AB = [A\text{col}_1(B) | \cdots | A\text{col}_n(B)]}$

$$(AB)^T = \left[\begin{array}{c} (A\text{col}_1(B))^T \\ \hline (A\text{col}_2(B))^T \\ \hline \vdots \\ \hline (A\text{col}_n(B))^T \end{array}\right]$$

$$B^T A^T = \left[\begin{array}{c} (\text{col}_1(B))^T A^T \\ \hline (\text{col}_2(B))^T A^T \\ \hline \vdots \\ \hline (\text{col}_n(B))^T A^T \end{array}\right] = \left[\begin{array}{c} \text{row}_1(B^T) A^T \\ \hline \text{row}_2(B^T) A^T \\ \hline \vdots \\ \hline \text{row}_n(B^T) A^T \end{array}\right]$$

$$\left[\begin{array}{c} \text{row}_1(B^T) \\ \hline \text{row}_2(B^T) \\ \hline \vdots \\ \hline \text{row}_n(B^T) \end{array}\right] A^T = \left[\begin{array}{c} \text{row}_1(B^T) A^T \\ \hline \text{row}_2(B^T) A^T \\ \hline \vdots \\ \hline \text{row}_n(B^T) A^T \end{array}\right] \quad \therefore \quad \boxed{\left[\begin{array}{c} W_1 \\ \hline W_2 \\ \hline \vdots \\ \hline W_m \end{array}\right] A = \left[\begin{array}{c} W_1 A \\ \hline W_2 A \\ \hline \vdots \\ \hline W_m A \end{array}\right]}$$

row-by-row multiplication rule $\longrightarrow$ for row vectors $W_1, \dots, W_m$ and $A \in \mathbb{R}^{m\times n}$

Def/ Let $E_{rs} \in \mathbb{R}^{m\times n}$ be defined by

$$(E_{rs})_{ij} = \delta_{ri}\delta_{sj} = \begin{cases} 1 & \text{if } i=r \text{ and } j=s \\ 0 & \text{if } i\neq r \text{ or } j\neq s \end{cases}$$

We call E_{rs} the (r,s)-th matrix-unit

We can build any matrix with matrix units of the appropriate size.

$\boxed{E1}$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ A_{21} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & A_{22} \end{bmatrix}$$

$$= A_{11}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_{11}} + A_{12}\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_{12}} + A_{21}\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_{21}} + A_{22}\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_{22}}$$

Thm/ If $A \in \mathbb{R}^{m\times n}$ and $E_{ij} \in \mathbb{R}^{m\times n}$ then $A = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}E_{ij}$

<u>Proof</u>:

$$\left(\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}E_{ij}\right)_{kl} = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}(E_{ij})_{kl}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}\,\delta_{ik}\delta_{jl}$$

$$= A_{kl}$$

$$\therefore\ A = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}E_{ij}$$

by the calculation above, they share same components. //

Th$^{\underline{m}}$/ Suppose $\mathbb{R}^{m\times n}$ contains A and $v \in \mathbb{R}^n$ then

$$v = \sum_{i=1}^{n} v_i e_i$$

$$A = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij} E_{ij}$$

$$\bar{e}_i^T A = row_i(A)$$

$$Ae_j = col_j(A)$$

$$A_{ij} = \bar{e}_i^T A e_j$$

$$E_{ij} E_{kl} = \delta_{jk} E_{il}$$

$$E_{ij} = \bar{e}_i e_j^T \qquad \bar{e}_i \in \mathbb{R}^m = \mathbb{R}^{m\times 1}, \; e_j \in \mathbb{R}^n = \mathbb{R}^{n\times 1}$$

$$e_i^T e_j = e_i \bullet e_j = \delta_{ij}$$

E2

$$\bar{e}_1 e_2^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}[0, 1, 0] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = E_{12}$$

E5

$$[1, 0, 0]\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 2$$

E3

$$\underbrace{(\bar{e}_i e_j^T)_{kl} = (\bar{e}_i)_k (e_j^T)_l = \delta_{ik}\delta_{jl} = (E_{ij})_{kl}}_{\therefore\; E_{ij} = \bar{e}_i e_j^T}$$

E4

$$E_{ij} E_{kl} = e_i e_j^T e_k e_l^T = e_i \delta_{jk} e_l^T = \delta_{jk} e_i e_l^T = \delta_{jk} E_{il}$$

$\boxed{E6}$

$$A E_{ij} = A e_i e_j^T = col_i(A)\, e_j^T = \Big[\, 0 \,\Big|\, \cdots \,\Big|\, \underbrace{col_i(A)}_{j^{\text{th}}\text{ column.}} \,\Big|\, \cdots \,\Big|\, 0 \,\Big]$$

$$\underline{col_k(AE_{ij}) = \delta_{kj}\, col_i(A)}.$$

$\boxed{E7}$

$$E_{ij} A = e_i e_j^T A = e_i\, row_j(A) = \begin{array}{c} \\ \\ \\ i^{\text{th}}\text{ row} \\ \\ \\ \\ \end{array}\!\!\!\!\begin{bmatrix} 0 \\ \vdots \\ 0 \\ row_j(A) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\underline{row_k(E_{ij}A) = \delta_{ki}\, row_j(A)}.$$

Def^{n}/ $A \in \mathbb{R}^{n\times n}$ is strictly upper triangular if $A_{ij} = 0$ for all $i \geq j$

Let A, B be strictly upper triangular,

$$A = \sum_{i<j} A_{ij} E_{ij} \quad \text{and} \quad B = \sum_{k<l} B_{kl} E_{kl}$$

$$\begin{aligned} AB &= \sum_{i<j} \sum_{k<l} A_{ij} B_{kl} E_{ij} E_{kl} \\ &= \sum_{i<j} \sum_{k<l} A_{ij} B_{kl}\, \delta_{jk} E_{il} \\ &= \sum_{i<j<l} A_{ij} B_{jl} E_{il} \end{aligned}$$

$$A = \begin{bmatrix} 0 & A_{12} & A_{13} \\ 0 & 0 & A_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

CLAIM

$A^n = 0$

$$A^2 = \sum_{i<j<l} A_{ij} A_{jl} E_{il}$$

$$A^2 = \begin{bmatrix} 0 & 0 & A_{12}A_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$