Lecture 6 : Matrices with a particular set of skills

We've studied a variety of special matrices, let's review a bit and I'll give some explicit examples,

- $I_n = [e_1 | e_2 | \cdots | e_n] = \text{Diag}(1, 1, \dots, 1)$ the <u>identity matrix</u>

  $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- $E_{ij} \in \mathbb{R}^{m \times n}$ has $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ $\quad E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{2 \times 3}$

- $A^T = A$ symmetric
- $A^T = -A$ antisymmetric
- upper triangular $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$, lower-$\Delta$ $\begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix}$
- strictly upper triangular $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

- $N_k \in \mathbb{R}^{k \times k}$ has 1 on all the superdiagonal

  $(N_k)_{12} = (N_k)_{23} = \cdots = (N_k)_{k-1,k} = 1$

  $$\underbrace{N_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad N_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{the nilpotent matrices}}$$

  orders 2, 3 and 4 respective

- $\boxed{A \in \mathbb{R}^{n \times n} \text{ is a } \underline{\text{square matrix}}. \text{ If } A \text{ is } \underline{\text{invertible}} \text{ then } \exists A^{-1} \in \mathbb{R}^{n \times n} \text{ s.t. } AA^{-1} = I = A^{-1}A}$

BLOCK MULTIPLICATION

We've seen a couple special cases of block multiplication with $A[B_1|B_2|\cdots|B_n] = [AB_1|AB_2|\cdots|AB_n]$. More generally, if we can parse a given matrix into submatrices and another matrix has a compatible decomposition then multiplication of such matrices can be done by treating the blocks as if they were #'s. I'll give examples,

E1

$$\boxed{\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]\left[\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right] = \left[\begin{array}{c|c} AX+BZ & AY+BW \\ \hline CX+DZ & CY+DW \end{array}\right]}$$

Here $A, B, C, D, X, Y, Z, W \in \mathbb{R}^{n\times n}$ whereas

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \left[\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right] \in \mathbb{R}^{2n\times 2n}$$

this is an example of "block" multiplication where A, B, C, D etc are the "blocks"

E2 $I_4 = \left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & I_2 \end{array}\right]$ nice shortcut for block-diagonal

$$\left[\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 7 & 8 \end{array}\right]^{-1} = \left[\begin{array}{c|c} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1} & 0 \\ \hline 0 & \begin{bmatrix} 1 & 1 \\ 7 & 8 \end{bmatrix}^{-1} \end{array}\right] = \left[\begin{array}{cc|cc} 2 & -1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ \hline 0 & 0 & 8 & -1 \\ 0 & 0 & -7 & 1 \end{array}\right]$$

doubt me? Let's check,

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 7 & 8 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 0 & 0 & 8 & -1 \\ 0 & 0 & -7 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \checkmark$$

Remark: not all big matrices allow this trick, but it's sure nice when they do.

E3

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \\ A_{31}B_{11} + A_{32}B_{21} \end{bmatrix}$$

Here $A_{11} \in \mathbb{R}^{m_1 \times n_1}$, $A_{21} \in \mathbb{R}^{m_2 \times n_1}$, $A_{31} \in \mathbb{R}^{m_3 \times n_1}$ | $A_{12} \in \mathbb{R}^{m_1 \times n_2}$, $A_{22} \in \mathbb{R}^{m_2 \times n_2}$, $A_{32} \in \mathbb{R}^{m_3 \times n_2}$ | $B_{11} \in \mathbb{R}^{n_1 \times p}$, $B_{21} \in \mathbb{R}^{n_2 \times p}$

$$\big[(m_1 + m_2 + m_3) \times (n_1 + n_2)\big]\big[(n_1 + n_2) \times p\big] \;:\; (m_1 + m_2 + m_3) \times p$$

In practice we're usually most interested in the case the matrix is square and all the blocks are square. Even better, most blocks are zero

Def/ For $A_1 \in \mathbb{R}^{m_1 \times m_1}$ and $A_2 \in \mathbb{R}^{m_2 \times m_2}$ we define $A_1 \oplus A_2 = \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right] \in \mathbb{R}^{(m_1 + m_2) \times (m_1 + m_2)}$

Likewise, for $A_1, A_2, \ldots, A_k$ all square,

$$A_1 \oplus A_2 \oplus \cdots \oplus A_k = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{bmatrix}$$

(all off-block entries are zero)

If $M = A_1 \oplus A_2 \oplus \cdots \oplus A_k$ for some set of square matrices $A_1, A_2, \ldots, A_k$ then we say M is in <u>block</u> – <u>diagonal</u> <u>form</u>.

<u>Remark</u>: $1 \oplus 2 \oplus 3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \text{Diag}(1,2,3)$ is simplest case, plain old diagonal matrix.

E4 $I_2 \oplus N_3 = \left[\begin{array}{c|c} I_2 & \\ \hline & N_3 \end{array}\right] = \left[\begin{array}{cc|ccc} 1 & 0 & & & \\ 0 & 1 & & & \\ \hline & & 0 & 1 & 0 \\ & & 0 & 0 & 1 \\ & & 0 & 0 & 0 \end{array}\right]$

> **Th^m/** Given block diagonal matrices A and B with matching dimension blocks,
>
> $$(A_1 \oplus A_2 \oplus \cdots \oplus A_k)(B_1 \oplus B_2 \oplus \cdots \oplus B_k) = (A_1B_1) \oplus (A_2B_2) \oplus \cdots \oplus (A_kB_k)$$
>
> Furthermore,
>
> $$(A_1 \oplus A_2 \oplus \cdots \oplus A_k)^m = (A_1^m) \oplus (A_2^m) \oplus \cdots \oplus (A_k^m)$$

E5 $(3I \oplus N_2 \oplus N_3)^2 = (3I)^2 \oplus (N_2^2) \oplus (N_3)^2$

$= (9I) \oplus (0) \oplus \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

E6 $(a \oplus b \oplus c)^2 = a^2 \oplus b^2 \oplus c^2$

$\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} = \begin{pmatrix} a^2 & & \\ & b^2 & \\ & & c^2 \end{pmatrix} = a^2 \oplus b^2 \oplus c^2$

E7 Given A has inverse A^{-1} $(n \times n)$ and B has inverse B^{-1} notice (B $m \times m$)

$(A \oplus B)(A^{-1} \oplus B^{-1}) = (AA^{-1}) \oplus (BB^{-1}) = I_n \oplus I_m$

$\underline{(A \oplus B)^{-1} = A^{-1} \oplus B^{-1}}$

ELEMENTARY MATRICES & ROW REDUCTION

Given an $(m \times n)$ matrix A which transforms to R under an elementary row operation we can relate A and R by LEFT multiplication by an appropriate $(m \times m)$ matrix E which we call an <u>ELEMENTARY MATRIX</u> with $R = EA$

$\boxed{E8}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} -3 & -3 & -3 \\ 4 & 5 & 6 \end{bmatrix} = R$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = E$$

$$EA = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -3 & -3 & -3 \\ 4 & 5 & 6 \end{bmatrix}$$

$\boxed{E9}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} = R$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = E$$

$$EA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} = R$$

Thm/ If $A \xrightarrow{r} R$ where r is a elementary row operation then if E is given by $I \xrightarrow{r} E$ then

$$R = EA$$

$\boxed{E10}$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[1]{r_2 - r_3} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[2]{r_3 - r_1} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[3]{r_1 \leftrightarrow r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[4]{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

1: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 - r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = E_1$

2: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_3 - r_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = E_2$

3: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3$

4: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_4$

$$\underbrace{E_4 E_3 E_2 E_1}_{E} A = I \longrightarrow \boxed{A^{-1} = E_4 E_3 E_2 E_1}$$

$$E = E_4 E_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = E_4 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(Remark: we do find a better way to calculate A^{-1} in an upcoming LECTURE)

$$\underbrace{\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{E}\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} \qquad \therefore \boxed{A^{-1} = E}$$