Lecture 7: Elementary Matrices & the CCP

The CCP or Column Correspondence Property is also sometimes called the linear correspondence. We'll derive it via the elementary matrices we introduced in Lecture 6.

$\boxed{E1}$

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 - 3r_2} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = R$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 - 3r_2} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E$$

$$EA = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 & 5 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = R$$

Every elementary matrix is invertible and its inverse matrix is also an elementary matrix,

$\boxed{E2}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 + 2r_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{3r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 - 2r_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1^{-1}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\frac{1}{3} E]{(\frac{1}{3}) r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2^{-1}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_3^{-1}$$

(we can verify $E_1 E_1^{-1} = I$, $E_2 E_2^{-1} = I$, $E_3 E_3^{-1} = I$)

$\boxed{\text{Th}^{\underline{m}}/ \text{ If } A \text{ and } B \text{ are both invertible square matrices then } (AB)^{-1} = B^{-1}A^{-1}. \text{ Moreover, if } A_1, A_2, \dots, A_m \text{ are invertible then } (A_1A_2\cdots A_m)^{-1} = A_m^{-1}\cdots A_2^{-1}A_1^{-1}}$

<u>Proof</u>: Suppose $A, B, A_1, \dots, A_m$ are all invertible,

$$(B^{-1}A^{-1})(AB) = B^{-1}IB = B^{-1}B = I$$
$$(AB)(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I$$

thus $(AB)^{-1} = B^{-1}A^{-1}$ as claimed. Suppose inductively $(A_1A_2\cdots A_{m-1})^{-1} = A_{m-1}^{-1}\cdots A_2^{-1}A_1^{-1}$ for some $m \geq 2$ and observe for $A = A_1A_2\cdots A_{m-1}$ and $B = A_m$ we have

$$AB = A_1A_2\cdots A_{m-1}A_m$$

thus, by the induction hypothesis and $(AB)^{-1} = B^{-1}A^{-1}$,

$$(A_1\cdots A_m)^{-1} = A_m^{-1}A_{m-1}^{-1}\cdots A_2^{-1}A_1^{-1}$$

Then the $\text{Th}^{\underline{m}}$ follows by proof by mathematical induction. //

E3 Let me spare us #'s and describe a row-reduction schematically,

$$A \xrightarrow{①} R_1 \xrightarrow{②} R_2 \xrightarrow{③} R_3 \cdots \xrightarrow{ⓜ} \text{rref}(A)$$

$$R_1 = E_1A$$
$$R_2 = E_2R_1 = E_2E_1A$$
$$R_3 = E_3R_2 = E_3E_2E_1A \quad \cdots \quad \boxed{\text{rref}(A) = \underbrace{E_m\cdots E_3E_2E_1}_{E = E_m\cdots E_3E_2E_1}A}$$

Observe $E^{-1} = E_1^{-1}E_2^{-1}\cdots E_m^{-1}$

$$\text{rref}(A) = EA \Rightarrow \underline{A = E^{-1}\text{rref}(A)}.$$

$\boxed{\text{}}$

Th^m (CCP) Suppose A and R are row-equivalent matrices in the sense that a finite sequence of row operations transforms A into R. Then any linear dependence of the columns of A will also be a linear dependence of the columns of R

<u>Proof</u>: Suppose $R = E_m \cdots E_2 E_1 A$ where $E_1, \dots, E_m$ are elementary matrices then $E = E_m \cdots E_2 E_1$ is an invertible matrix; $R = EA$ and $A = E^{-1}R$. Suppose $\exists\, c_1, c_2, \dots, c_n \in \mathbb{R}$ for which

$$\boxed{c_1 \,col_1(A) + c_2\, col_2(A) + \cdots + c_n\, col_n(A) = 0} \; *$$

Notice $E\,col_i(A) = col_i(EA)$ by the column-by-column rule; $E[col_1(A)|col_2(A)|\cdots|col_n(A)] = [E col_1(A)|E col_2(A)|\cdots|E col_n(A)]$.

Thus, multiplying $*$ we obtain:

$$c_1 E\,col_1(A) + c_2 E\,col_2(A) + \cdots + c_n E\,col_n(A) = 0$$

$$\Rightarrow c_1\, col_1(EA) + c_2\, col_2(EA) + \cdots + c_n\, col_n(EA) = 0$$

$$\Rightarrow \boxed{c_1\, col_1(R) + c_2\, col_2(R) + \cdots + c_n\, col_n(R) = 0} \; **$$

Thus any linear dependence of columns of A must also be found among columns of R. Likewise, if $\exists\, b_1, b_2, \dots, b_n \in \mathbb{R}$ for which $\sum_{i=1}^{n} b_i\, col_i(R) = 0$

then $E^{-1}\left(\sum_{i=1}^{n} b_i\, col_i(R)\right) = \sum_{i=1}^{n} b_i\, col_i(E^{-1}R) = \sum_{i=1}^{n} b_i\, col_i(A) = 0.$

Thus any linear dep. of columns of R must be found amongst the columns of A.

Corollary: columns of A are LI $\Leftrightarrow$ columns of R are LI whenever A and R are row-equivalent matrices.

Remark: A is row-equivalent to $\text{rref}(A)$ and it is most obvious to simply **see** any linear dep. amongst columns in $\text{rref}(A)$.

E4

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & 4 \\ 4 & 1 & 1 & 4 \end{bmatrix} \sim \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$col_4 = col_1 - col_2 + col_3$$

(true for both A and $\text{rref}(A)$) $\quad \{col_1, col_2, col_3\}$ LI

E5

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \sim \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{bmatrix}$$

$$\left.\begin{aligned} col_4 &= -2col_1 - 3col_2 - 2col_3 \\ col_5 &= -3col_1 - 3col_2 - 4col_3 \\ col_6 &= -col_1 - col_2 - col_3 \end{aligned}\right\} \text{by inspection of}$$

$\{col_1, col_2, col_3\}$ LI

E6

$$A = \begin{bmatrix} 1 & 2 & 5 & 6 \\ 2 & 4 & 5 & 7 \end{bmatrix} \sim \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\left.\begin{aligned} col_2 &= 2col_1 \\ col_4 &= col_1 + col_3 \end{aligned}\right\} \text{by inspection of}$$

$\{col_1, col_3\}$ LI

> **Th$^{\underline{m}}$/** Given $A \in \mathbb{R}^{m \times n}$ we find the pivot columns of A form a LI set while every non-pivotal column can be uniquely expressed as ~~a~~ linear combination of the pivot columns.

Proof: the claim of the Th$^{\underline{m}}$ is clearly true for $\text{rref}(A)$ and hence by the CCP it likewise holds for A. //

REVERSE ENGINEERING WITH CCP

E7 Suppose $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Given $\text{col}_1(A) = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$ and $\text{col}_2(A) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\text{col}_5(A) = \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}$

find A.

By CCP and inspection of given rref(A),

$$\text{col}_3(A) = 3\,\text{col}_1(A) + 2\,\text{col}_2(A) = 3\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 23 \end{bmatrix}$$

$$\text{col}_4(A) = \text{col}_1(A) - \text{col}_2(A) = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

Therefore,

$$\boxed{A = \begin{bmatrix} 1 & 0 & 3 & 1 & 7 \\ 3 & 1 & 11 & 2 & 7 \\ 7 & 1 & 23 & 6 & 7 \end{bmatrix}}$$