LECTURE 8 : INVERTIBLE MATRICES, PROPERTIES & CALCULATION

We have introduced invertible matrices in earlier lectures. We said $A \in \mathbb{R}^{n\times n}$ is invertible with <u>inverse matrix</u> A^{-1} provided $A^{-1}A = I_n = [e_1|e_2|\cdots|e_n] = AA^{-1}$. We also saw that there is a formula for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Also, we derived $(A_1A_2\cdots A_m)^{-1} = A_m^{-1}\cdots A_2^{-1}A_1^{-1}$ in the case that $A_1, A_2, \dots, A_m$ are invertible matrices. Block mult. also gave us an extension $(A_1 \oplus A_2 \oplus \cdots \oplus A_m)^{-1} = A_1^{-1} \oplus A_2^{-1} \oplus \cdots \oplus A_m^{-1}$. We now give the further theory of invertible matrices and describe how to calculate A^{-1} when A is not so simple. Equally important, we seek to characterize when a square matrix A is not invertible.

Th^m ① : Given $A \in \mathbb{R}^{n\times n}$, the solution of $Ax = 0$ is unique if and only if A^{-1} exists.

Proof: $\Leftarrow$ Suppose A^{-1} exists. If $Ax = 0$ then $A^{-1}Ax = A^{-1}0$ thus $Ix = 0$ hence $x = 0$ (this is the unique sol^n of $Ax = 0$).

$\Rightarrow$ Suppose the solution of $Ax = 0$ is unique. Notice $A(0) = 0$ then $x = 0$ is the only solution of $Ax = 0$. Hence $\text{rref}[A|0] = [I_n|0]$ and so $\exists\ E = E_m \cdots E_2E_1$ with $\underline{EA = I_n}_{\,*}$ and $E_1, E_2, \dots, E_m$ are elementary matrices. Then multiply $EA = I_n$ by $E^{-1} = E_1^{-1}E_2^{-1}\cdots E_m^{-1}$ on the left to obtain

$$E^{-1}EA = E^{-1}I_n \Rightarrow A = E^{-1}$$

then $\underline{AE = E^{-1}E = I}_{\,**}$ $\therefore$ $\underline{A^{-1} = E}$ by $*$ and $**$

Remark: the proof above reveals several facts about A^{-1} we record in our next theorem ↴

Th$^{\underline{m}}$ 2 Given $A \in \mathbb{R}^{n\times n}$,

(a.) A^{-1} exists $\Leftrightarrow$ $rref(A) = I_n$

(b.) A^{-1} exists $\Leftrightarrow$ A is product of elementary matrices

(c.) A^{-1} exists $\Leftrightarrow$ $\exists\, B \in \mathbb{R}^{n\times n}$ with $BA = I_n$

(d.) A^{-1} exists $\Leftrightarrow$ $\exists\, B \in \mathbb{R}^{n\times n}$ with $AB = I_n$

(e.) A^{-1} exists $\Leftrightarrow$ for any $b \in \mathbb{R}^n$, $rref[A|b] = [I_n|x]$

(f.) A^{-1} exists $\Leftrightarrow$ $Ax = b$ has unique solution for any given inhomogeneous term $b \in \mathbb{R}^n$

__Proof:__ parts a., b., e. follow from proof of Th$^{\underline{m}}$ 1. The proof of (f.) I'll give now. If A^{-1} exists and $Ax = b$ then $A^{-1}Ax = A^{-1}b \Rightarrow Ix = A^{-1}b \Rightarrow \underline{x = A^{-1}b}$. Conversely, if x is the unique sol$^{\underline{n}}$ of $Ax = b$ for a given $b \in \mathbb{R}^n$ then by the row-reduction technique $rref[A|b] = [I_n|x]$ thus $EA = I_n$ and, by proof in Th$^{\underline{m}}$ 1, $AE = I_n$ hence $A^{-1} = E$ and that completes the argument for (f.).

Proof of (c.): the forward $\Rightarrow$ is given by def$^{\underline{n}}$ of A^{-1}. Suppose $\exists\, B \in \mathbb{R}^{n\times n}$ such that $BA = I_n$. It remains to show $AB = I$. Suppose $Ax = 0$ then $BAx = B(0) \Rightarrow Ix = 0 \Rightarrow x = 0$ thus $Ax = 0$ has only the $x = 0$ solution hence by Th$^{\underline{m}}$ 1 we have A^{-1} exists hence $AB = I$ since $BA = I$ (A^{-1} is the unique matrix B for which $AB = I = BA$)

Proof of (d.): $\Rightarrow$ is immediately true from def$^{\underline{n}}$ of A^{-1}. Conversely, if $\exists\, B \in \mathbb{R}^{n\times n}$ and $AB = I_n$ then it remains to show why $BA = I_n$. Suppose $Bx = 0$ then $ABx = A0 \Rightarrow Ix = 0 \Rightarrow x = 0$ thus B^{-1} exists by Th$^{\underline{m}}$ 1 and since $AB = I$ we must have $BA = I$ as well. Indeed, $A^{-1} = B$ and $B^{-1} = (A^{-1})^{-1} = A$. //

Th$^{\underline{m}}$ ③ Let $A \in \mathbb{R}^{n\times n}$.

(a.) If A^{-1} exists then $(A^{-1})^{-1} = A$

(b.) If A^{-1} exists and $c \in \mathbb{R}$ is not zero then $(cA)^{-1} = \frac{1}{c}A^{-1}$.

(c.) A^{-1} exists $\Longleftrightarrow$ $(A^T)^{-1}$ exists

(d.) If A^{-1} exists then $(A^T)^{-1} = (A^{-1})^T$

Proof: we proved (a.) in our arguments for Th$^{\underline{m}}$ ②. Suppose A^{-1} exists and $c \neq 0$ then

$$(cA)\left(\tfrac{1}{c}A^{-1}\right) = \tfrac{c}{c}AA^{-1} = 1 \cdot I_n = I_n$$

thus $(cA)^{-1} = \frac{1}{c}A^{-1}$ by Th$^{\underline{m}}$ ② part (d.). Suppose A^{-1} exists then $A^{-1}A = I$ then by socks-shoes rule for transpose,

$$A^T(A^{-1})^T = I^T = I \quad \text{thus } (A^T)^{-1} = (A^{-1})^T.$$

I'll leave the proof that $(A^T)^{-1}$ exists $\Rightarrow A^{-1}$ exists to the reader.

Def$^{\underline{n}}$/ A matrix $R \in \mathbb{R}^{n\times n}$ such that $R^TR = I_n$ is known as an orthogonal matrix and we write $R \in O(n)$ or $\underbrace{R \in O(n,\mathbb{R}) \text{ or } R \in O_n(\mathbb{R})}$

(sorry, I tend to forget which I'm using in a given course)

E1 If $A \in O(n)$ then since $A^TA = I$ we find $\underline{A^{-1} = A^T}$.

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in O(2) \text{ and } A^TA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

ALGORITHM TO CALCULATE A^{-1} (OR NOT)

Consider A an $(n \times n)$ matrix and let $A^{-1} = [v_1 | v_2 | \cdots | v_n]$

we need to solve $AA^{-1} = I_n$ hence

$$A[v_1 | v_2 | \cdots | v_n] = [e_1 | e_2 | \cdots | e_n]$$

$$[Av_1 | Av_2 | \cdots | Av_n] = [e_1 | e_2 | \cdots | e_n]$$

$$\underbrace{Av_j = e_j \text{ for } j = 1, 2, \ldots}_{}, n$$

we face n-systems of n-eq^{ns} where each coeff matrix is A.

$$\text{rref}(A | e_j) = [I_n | v_j]$$

therefore, <u>if A^{-1} exists</u>

$$\boxed{\begin{array}{l} \text{Th}^{m}/\ \text{rref}(A | I_n) = [I_n | A^{-1}]. \text{ Moreover,} \\ \text{if rref}(A | I_n) \neq [I_n | B] \text{ then } A^{-1}\ \underbrace{\text{d.n.e.}}_{\text{"does not exist"}} \end{array}}$$

LINEAR ALGEBRA TOOLKIT

Transforming a matrix to reduced row echelon form

v. 1.25

TRANSFORMING MATRIX TO THE REDUCED ROW ECHELON FORM

Row Operation 1:	$\begin{array}{cc} A & I \\ \boxed{\begin{matrix}1&4&5\\2&4&6\\3&4&7\end{matrix}} & \boxed{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}\end{array}$	add **-2** times the 1st row to the 2nd row	$\begin{bmatrix}1&4&5&1&0&0\\0&-4&-4&-2&1&0\\3&4&7&0&0&1\end{bmatrix}$
Row Operation 2:	$\begin{bmatrix}1&4&5&1&0&0\\0&-4&-4&-2&1&0\\3&4&7&0&0&1\end{bmatrix}$	add **-3** times the 1st row to the 3rd row	$\begin{bmatrix}1&4&5&1&0&0\\0&-4&-4&-2&1&0\\0&-8&-8&-3&0&1\end{bmatrix}$
Row Operation 3:	$\begin{bmatrix}1&4&5&1&0&0\\0&-4&-4&-2&1&0\\0&-8&-8&-3&0&1\end{bmatrix}$	multiply the 2nd row by **-1/4**	$\begin{bmatrix}1&4&5&1&0&0\\0&1&1&\frac{1}{2}&\frac{-1}{4}&0\\0&-8&-8&-3&0&1\end{bmatrix}$
Row Operation 4:	$\begin{bmatrix}1&4&5&1&0&0\\0&1&1&\frac{1}{2}&\frac{-1}{4}&0\\0&-8&-8&-3&0&1\end{bmatrix}$	add **8** times the 2nd row to the 3rd row	$\begin{bmatrix}1&4&5&1&0&0\\0&1&1&\frac{1}{2}&\frac{-1}{4}&0\\0&0&0&1&-2&1\end{bmatrix}$
Row Operation 5:	$\begin{bmatrix}1&4&5&1&0&0\\0&1&1&\frac{1}{2}&\frac{-1}{4}&0\\0&0&0&1&-2&1\end{bmatrix}$	add **-1/2** times the 3rd row to the 2nd row	$\begin{bmatrix}1&4&5&1&0&0\\0&1&1&0&\frac{3}{4}&\frac{-1}{2}\\0&0&0&1&-2&1\end{bmatrix}$
Row Operation 6:	$\begin{bmatrix}1&4&5&1&0&0\\0&1&1&0&\frac{3}{4}&\frac{-1}{2}\\0&0&0&1&-2&1\end{bmatrix}$	add **-1** times the 3rd row to the 1st row	$\begin{bmatrix}1&4&5&0&2&-1\\0&1&1&0&\frac{3}{4}&\frac{-1}{2}\\0&0&0&1&-2&1\end{bmatrix}$
Row Operation 7:	$\begin{bmatrix}1&4&5&0&2&-1\\0&1&1&0&\frac{3}{4}&\frac{-1}{2}\\0&0&0&1&-2&1\end{bmatrix}$	add **-4** times the 2nd row to the 1st row	$\begin{bmatrix}1&0&1&0&-1&1\\0&1&1&0&\frac{3}{4}&\frac{-1}{2}\\0&0&0&1&-2&1\end{bmatrix}$

Thus rref $\left[\underbrace{\begin{matrix}1&4&5\\2&4&6\\3&4&7\end{matrix}}_{A}\middle|\underbrace{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}_{I}\right] \neq [I_3 | *] \therefore \begin{bmatrix}1&4&5\\2&4&6\\3&4&7\end{bmatrix}^{-1}$ does not exist.

E3

LINEAR ALGEBRA TOOLKIT

Transforming a matrix to reduced row echelon form

v. 1.25

TRANSFORMING MATRIX TO THE REDUCED ROW ECHELON FORM

Row Operation 1:

$$\begin{array}{cc} A & I \\ \left[\begin{matrix} 1 & 2 & 3 \\ 2 & 2 & 7 \\ 3 & 4 & -9 \end{matrix}\right. & \left.\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right] \end{array}$$

add **-2** times the 1st row to the 2nd row

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & 1 & -2 & 1 & 0 \\ 3 & 4 & -9 & 0 & 0 & 1 \end{bmatrix}$$

Row Operation 2:

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & 1 & -2 & 1 & 0 \\ 3 & 4 & -9 & 0 & 0 & 1 \end{bmatrix}$$

add **-3** times the 1st row to the 3rd row

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & 1 & -2 & 1 & 0 \\ 0 & -2 & -18 & -3 & 0 & 1 \end{bmatrix}$$

Row Operation 3:

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & 1 & -2 & 1 & 0 \\ 0 & -2 & -18 & -3 & 0 & 1 \end{bmatrix}$$

multiply the 2nd row by **-1/2**

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} & 1 & \frac{-1}{2} & 0 \\ 0 & -2 & -18 & -3 & 0 & 1 \end{bmatrix}$$

Row Operation 4:

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} & 1 & \frac{-1}{2} & 0 \\ 0 & -2 & -18 & -3 & 0 & 1 \end{bmatrix}$$

add **2** times the 2nd row to the 3rd row

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & -19 & -1 & -1 & 1 \end{bmatrix}$$

Row Operation 5:

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & -19 & -1 & -1 & 1 \end{bmatrix}$$

multiply the 3rd row by **-1/19**

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{19} & \frac{1}{19} & \frac{-1}{19} \end{bmatrix}$$

Row Operation 6:

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{19} & \frac{1}{19} & \frac{-1}{19} \end{bmatrix}$$

add **1/2** times the 3rd row to the 2nd row

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{39}{38} & \frac{-9}{19} & \frac{-1}{38} \\ 0 & 0 & 1 & \frac{1}{19} & \frac{1}{19} & \frac{-1}{19} \end{bmatrix}$$

E3 continued

Row Operation 7:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{39}{38} & \frac{-9}{19} & \frac{-1}{38} \\ 0 & 0 & 1 & \frac{1}{19} & \frac{1}{19} & \frac{-1}{19} \end{array}\right]$$

add **-3** times the 3rd row to the 1st row

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{16}{19} & \frac{-3}{19} & \frac{3}{19} \\ 0 & 1 & 0 & \frac{39}{38} & \frac{-9}{19} & \frac{-1}{38} \\ 0 & 0 & 1 & \frac{1}{19} & \frac{1}{19} & \frac{-1}{19} \end{array}\right]$$

Row Operation 8:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{16}{19} & \frac{-3}{19} & \frac{3}{19} \\ 0 & 1 & 0 & \frac{39}{38} & \frac{-9}{19} & \frac{-1}{38} \\ 0 & 0 & 1 & \frac{1}{19} & \frac{1}{19} & \frac{-1}{19} \end{array}\right]$$

add **-2** times the 2nd row to the 1st row

$$\left[\;\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \;\middle|\; \underbrace{\begin{matrix} \frac{-23}{19} & \frac{15}{19} & \frac{4}{19} \\ \frac{39}{38} & \frac{-9}{19} & \frac{-1}{38} \\ \frac{1}{19} & \frac{1}{19} & \frac{-1}{19} \end{matrix}}_{A^{-1}}\;\right]$$

Therefore,

$$\boxed{A^{-1} = \frac{1}{38}\begin{bmatrix} -46 & 30 & 8 \\ 39 & -18 & -1 \\ 2 & 2 & -2 \end{bmatrix}}$$

We can check our work,

$$A^{-1}A = \frac{1}{38}\begin{bmatrix} -46 & 30 & 8 \\ 39 & -18 & -1 \\ 2 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 7 \\ 3 & 4 & -9 \end{bmatrix}$$

$$= \frac{1}{38}\left[\begin{array}{c|c|c} -46+60+24 & 0 & 0 \\ \hline 0 & 78-36-4 & 0 \\ \hline 0 & 0 & 6+14+18 \end{array}\right]$$

$$= \frac{1}{38}\begin{bmatrix} 38 & 0 & 0 \\ 0 & 38 & 0 \\ 0 & 0 & 38 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

TRANSFORMING MATRIX TO THE REDUCED ROW ECHELON FORM

A I_4

Row Operation 1:

$$\begin{bmatrix} 6 & 4 & -2 & 0 & 1 & 0 & 0 & 0 \\ 8 & 8 & 8 & 8 & 0 & 1 & 0 & 0 \\ 4 & 0 & 2 & 2 & 0 & 0 & 1 & 0 \\ -4 & 6 & 2 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

multiply the 1st row by **1/6**

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 8 & 8 & 8 & 8 & 0 & 1 & 0 & 0 \\ 4 & 0 & 2 & 2 & 0 & 0 & 1 & 0 \\ -4 & 6 & 2 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Row Operation 2:

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 8 & 8 & 8 & 8 & 0 & 1 & 0 & 0 \\ 4 & 0 & 2 & 2 & 0 & 0 & 1 & 0 \\ -4 & 6 & 2 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

add **-8** times the 1st row to the 2nd row

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{8}{3} & \frac{32}{3} & 8 & \frac{-4}{3} & 1 & 0 & 0 \\ 4 & 0 & 2 & 2 & 0 & 0 & 1 & 0 \\ -4 & 6 & 2 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Row Operation 3:

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{8}{3} & \frac{32}{3} & 8 & \frac{-4}{3} & 1 & 0 & 0 \\ 4 & 0 & 2 & 2 & 0 & 0 & 1 & 0 \\ -4 & 6 & 2 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

add **-4** times the 1st row to the 3rd row

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{8}{3} & \frac{32}{3} & 8 & \frac{-4}{3} & 1 & 0 & 0 \\ 0 & \frac{-8}{3} & \frac{10}{3} & 2 & \frac{-2}{3} & 0 & 1 & 0 \\ -4 & 6 & 2 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Row Operation 4:

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{8}{3} & \frac{32}{3} & 8 & \frac{-4}{3} & 1 & 0 & 0 \\ 0 & \frac{-8}{3} & \frac{10}{3} & 2 & \frac{-2}{3} & 0 & 1 & 0 \\ -4 & 6 & 2 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

add **4** times the 1st row to the 4th row

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{8}{3} & \frac{32}{3} & 8 & \frac{-4}{3} & 1 & 0 & 0 \\ 0 & \frac{-8}{3} & \frac{10}{3} & 2 & \frac{-2}{3} & 0 & 1 & 0 \\ 0 & \frac{26}{3} & \frac{2}{3} & 2 & \frac{2}{3} & 0 & 0 & 1 \end{bmatrix}$$

Row Operation 5:

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{8}{3} & \frac{32}{3} & 8 & \frac{-4}{3} & 1 & 0 & 0 \\ 0 & \frac{-8}{3} & \frac{10}{3} & 2 & \frac{-2}{3} & 0 & 1 & 0 \\ 0 & \frac{26}{3} & \frac{2}{3} & 2 & \frac{2}{3} & 0 & 0 & 1 \end{bmatrix}$$

multiply the 2nd row by **3/8**

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 & \frac{-1}{2} & \frac{3}{8} & 0 & 0 \\ 0 & \frac{-8}{3} & \frac{10}{3} & 2 & \frac{-2}{3} & 0 & 1 & 0 \\ 0 & \frac{26}{3} & \frac{2}{3} & 2 & \frac{2}{3} & 0 & 0 & 1 \end{bmatrix}$$

Row Operation 6:

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 & \frac{-1}{2} & \frac{3}{8} & 0 & 0 \\ 0 & \frac{-8}{3} & \frac{10}{3} & 2 & \frac{-2}{3} & 0 & 1 & 0 \end{bmatrix}$$

add **8/3** times the 2nd row to the 3rd row

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 & \frac{-1}{2} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & 14 & 10 & -2 & 1 & 1 & 0 \\ 0 & 26 & 2 & 2 & 2 & 0 & 0 & 1 \end{bmatrix}$$

E4 continued

$$\begin{matrix} 0 & \frac{26}{3} & \frac{2}{3} & 2 & \frac{2}{3} & 0 & 0 & 1 \end{matrix}$$

$$\begin{matrix} & 3 & 3 & & 3 & & & \end{matrix}$$

Row Operation 7:

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 & \frac{-1}{2} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & 14 & 10 & -2 & 1 & 1 & 0 \\ 0 & \frac{26}{3} & \frac{2}{3} & 2 & \frac{2}{3} & 0 & 0 & 1 \end{bmatrix}$$

add **-26/3** times the 2nd row to the 4th row

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 & \frac{-1}{2} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & 14 & 10 & -2 & 1 & 1 & 0 \\ 0 & 0 & -34 & -24 & 5 & \frac{-13}{4} & 0 & 1 \end{bmatrix}$$

Row Operation 8:

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 & \frac{-1}{2} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & 14 & 10 & -2 & 1 & 1 & 0 \\ 0 & 0 & -34 & -24 & 5 & \frac{-13}{4} & 0 & 1 \end{bmatrix}$$

multiply the 3rd row by **1/14**

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 & \frac{-1}{2} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & 1 & \frac{5}{7} & \frac{-1}{7} & \frac{1}{14} & \frac{1}{14} & 0 \\ 0 & 0 & -34 & -24 & 5 & \frac{-13}{4} & 0 & 1 \end{bmatrix}$$

Row Operation 9:

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 & \frac{-1}{2} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & 1 & \frac{5}{7} & \frac{-1}{7} & \frac{1}{14} & \frac{1}{14} & 0 \\ 0 & 0 & -34 & -24 & 5 & \frac{-13}{4} & 0 & 1 \end{bmatrix}$$

add **34** times the 3rd row to the 4th row

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 & \frac{-1}{2} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & 1 & \frac{5}{7} & \frac{-1}{7} & \frac{1}{14} & \frac{1}{14} & 0 \\ 0 & 0 & 0 & \frac{2}{7} & \frac{1}{7} & \frac{-23}{28} & \frac{17}{7} & 1 \end{bmatrix}$$

Row Operation 10:

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 & \frac{-1}{2} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & 1 & \frac{5}{7} & \frac{-1}{7} & \frac{1}{14} & \frac{1}{14} & 0 \\ 0 & 0 & 0 & \frac{2}{7} & \frac{1}{7} & \frac{-23}{28} & \frac{17}{7} & 1 \end{bmatrix}$$

multiply the 4th row by **7/2**

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 & \frac{-1}{2} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & 1 & \frac{5}{7} & \frac{-1}{7} & \frac{1}{14} & \frac{1}{14} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{-23}{8} & \frac{17}{2} & \frac{7}{2} \end{bmatrix}$$

Row Operation 11:

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 & \frac{-1}{2} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & 1 & \frac{5}{7} & \frac{-1}{7} & \frac{1}{14} & \frac{1}{14} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{-23}{8} & \frac{17}{2} & \frac{7}{2} \end{bmatrix}$$

add **-5/7** times the 4th row to the 3rd row

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 & \frac{-1}{2} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{-1}{2} & \frac{17}{8} & -6 & \frac{-5}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{-23}{8} & \frac{17}{2} & \frac{7}{2} \end{bmatrix}$$

Row Operation 12:

$$\begin{matrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \end{matrix}$$

add **-3** times the 4th row to the 2nd row

$$\begin{matrix} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \end{matrix}$$

E4 continued

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 4 & 3 & \frac{-1}{2} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{-1}{2} & \frac{17}{8} & -6 & \frac{-5}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{-23}{8} & \frac{17}{2} & \frac{7}{2} \end{array}\right] \qquad \left[\begin{array}{cccc|cccc} 0 & 1 & 4 & 0 & -2 & 9 & \frac{-51}{2} & \frac{-21}{2} \\ 0 & 0 & 1 & 0 & \frac{-1}{2} & \frac{17}{8} & -6 & \frac{-5}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{-23}{8} & \frac{17}{2} & \frac{7}{2} \end{array}\right]$$

Row Operation 13: add **-4** times the 3rd row to the 2nd row

$$\left[\begin{array}{cccc|cccc} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & -2 & 9 & \frac{-51}{2} & \frac{-21}{2} \\ 0 & 0 & 1 & 0 & \frac{-1}{2} & \frac{17}{8} & -6 & \frac{-5}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{-23}{8} & \frac{17}{2} & \frac{7}{2} \end{array}\right] \qquad \left[\begin{array}{cccc|cccc} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{-3}{2} & \frac{-1}{2} \\ 0 & 0 & 1 & 0 & \frac{-1}{2} & \frac{17}{8} & -6 & \frac{-5}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{-23}{8} & \frac{17}{2} & \frac{7}{2} \end{array}\right]$$

Row Operation 14: add **1/3** times the 3rd row to the 1st row

$$\left[\begin{array}{cccc|cccc} 1 & \frac{2}{3} & \frac{-1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{-3}{2} & \frac{-1}{2} \\ 0 & 0 & 1 & 0 & \frac{-1}{2} & \frac{17}{8} & -6 & \frac{-5}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{-23}{8} & \frac{17}{2} & \frac{7}{2} \end{array}\right] \qquad \left[\begin{array}{cccc|cccc} 1 & \frac{2}{3} & 0 & 0 & 0 & \frac{17}{24} & -2 & \frac{-5}{6} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{-3}{2} & \frac{-1}{2} \\ 0 & 0 & 1 & 0 & \frac{-1}{2} & \frac{17}{8} & -6 & \frac{-5}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{-23}{8} & \frac{17}{2} & \frac{7}{2} \end{array}\right]$$

Row Operation 15: add **-2/3** times the 2nd row to the 1st row

$$\left[\begin{array}{cccc|cccc} 1 & \frac{2}{3} & 0 & 0 & 0 & \frac{17}{24} & -2 & \frac{-5}{6} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{-3}{2} & \frac{-1}{2} \\ 0 & 0 & 1 & 0 & \frac{-1}{2} & \frac{17}{8} & -6 & \frac{-5}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{-23}{8} & \frac{17}{2} & \frac{7}{2} \end{array}\right] \qquad \left[\underbrace{\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}}_{I_4} \middle| \underbrace{\begin{array}{cccc} 0 & \frac{3}{8} & -1 & \frac{-1}{2} \\ 0 & \frac{1}{2} & \frac{-3}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{17}{8} & -6 & \frac{-5}{2} \\ \frac{1}{2} & \frac{-23}{8} & \frac{17}{2} & \frac{7}{2} \end{array}}_{A^{-1}}\right]$$

Therefore,

$$\boxed{A^{-1} = \frac{1}{8}\begin{bmatrix} 0 & 3 & -8 & -4 \\ 0 & 4 & -12 & -4 \\ -4 & 17 & -48 & -40 \\ 4 & -23 & 68 & 28 \end{bmatrix}}$$

$$A = \begin{bmatrix} 6 & 4 & -2 & 0 \\ 8 & 8 & 8 & 8 \\ 4 & 0 & 2 & 2 \\ -4 & 6 & 2 & 2 \end{bmatrix}$$

can check $A^{-1}A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.