LECTURE 9: NULL SPACE, COLUMN SPACE, THE RANK-NULLITY Th$^{\underline{m}}$

In this lecture we examine how LI and spanning allow us to describe solution sets and we explore the subspaces which are naturally tied to a given matrix. We also begin the conversation about subspaces or affine subspaces with $\mathbb{R}^n$ by making the definition of basis and dimension

Def$^{\underline{n}}$/ A space, or subset W is said to have basis $\beta = \{w_1, w_2, \dots, w_m\}$ if β is LI and $\text{span}(\beta) = W$. In this case we say $\dim(W) = \#(\beta) = m$.

$\boxed{E1}$ $W_1 = \text{span}\{(1,1,0,0), (0,0,1,-1)\}$ is a two-dim'l space with basis $\beta = \{(1,1,0,0), (0,0,1,-1)\}$.

Naturally, we might wonder if we can write W in a different fashion, I mean, do we have to use span? I'll circle back to this question a bit later.

Remark: checking LI of set with two vectors is easy. We simply need to check $w_1 \neq c w_2$ for nonzero w_1, w_2 and we can conclude $\{w_1, w_2\}$ is LI.

$\boxed{E2}$ $W_2 = \text{span}\{(1,-1,1)\}$ is one-dim'l space with basis $\{(1,-1,1)\}$. This is a line.

$\boxed{E3}$ $W_3 = \text{span}\{(1,2,2), (0,1,-1)\}$ is two-dim'l space with basis $\{(1,2,2),(0,1,-1)\}$. This is a plane.

$\boxed{E4}$ $W_4 = \mathbb{R}^n = \text{span}\{\underbrace{e_1, e_2, \dots, e_n}_{\text{the standard basis}}\}$ ← n-dim'l space

$\boxed{E5}$ $\mathbb{R}^{m\times n} = \text{span}\{E_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$, $\dim(\mathbb{R}^{m\times n}) = mn$.

$\boxed{E6}$ $P_2(\mathbb{R}) = \text{span}\{1, x, x^2\}$, $\dim(P_2(\mathbb{R})) = 3$.

Two particularly important subspaces are given for a matrix $A \in \mathbb{R}^{m\times n}$, the column space as a subspace of $\mathbb{R}^m$ and the null space as a subspace of $\mathbb{R}^n$.

Def^n/ Let $A \in \mathbb{R}^{m\times n}$ then

(1.) $\text{Col}(A) = \text{span}\{\text{col}_1(A), \dots, \text{col}_n(A)\}$ and we define $\dim(\text{Col}(A)) = \text{rank}(A) = \text{rk}(A)$. The <u>rank of A</u> is the dimension of the column space.

(2.) $\text{Null}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$ is the null space of A and $\dim(\text{Null}(A)) = \text{nullity}(A) = \nu(A)$. The <u>nullity of A</u> is the dimension of null space.

We should review what the matrix-column product means in terms of linear combinations of columns for $A \in \mathbb{R}^{m\times n}$

$$\boxed{Ax = x_1\,\text{col}_1(A) + x_2\,\text{col}_2(A) + \cdots + x_n\,\text{col}(A)}$$

Let's look at a few examples, the CCP guides our logic,

E7 $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ find basis for $\text{Col}(A)$ & $\text{Null}(A)$.

Notice $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ thus $\text{col}_1(A)$ is related to the others by $\text{col}_2(A) = 2\,\text{col}_1(A)$, $\text{col}_3(A) = 3\,\text{col}_1(A)$. The <u>basis for $\text{Col}(A)$ is $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$</u> thus $\boxed{\text{rank}(A) = 1}$.

If $x \in \text{Null}(A)$ then $Ax = 0$ thus $x_1 + 2x_2 + 3x_3 = 0$ thus $x_1 = -2x_2 - 3x_3$ and so,

$$x = (x_1, x_2, x_3) = (-2x_2 - 3x_3, x_2, x_3)$$
$$= x_2(-2,1,0) + x_3(-3,0,1)$$

Thus $\text{Null}(A) = \text{span}\underbrace{\{(-2,1,0), (-3,0,1)\}}_{\text{BASIS FOR Null}(A)}$ thus $\boxed{\text{nullity}(A) = 2}$

$\boxed{E8}$ $$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{rref}(A)$$

We find the basis for column space is given by the pivot columns since $\{col_1(A), col_2(A)\}$ is LI and columns 3 & 4 are formed by $col_3(A) = col_1(A) + col_2(A)$ and $col_4 A = col_1 A - col_2 \mathbf{A}$.

$\boxed{\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}}$ is basis for $Col(A)$, $\boxed{\text{rank}(A) = 2}$

Notice $Ax = 0$ implies $\begin{aligned} x_1 &= -x_3 - x_4 \\ x_2 &= -x_3 + x_4 \end{aligned}$ $\left(\text{rref}[A|0] = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \right)$

$$x = (x_1, x_2, x_3, x_4) = (-x_3 - x_4, -x_3 + x_4, x_3, x_4)$$
$$= x_3(-1, -1, 1, 0) + x_4(-1, 1, 0, 1)$$

BASIS FOR $Null(A)$,

$\boxed{\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}}$ $\Rightarrow$ $\boxed{\text{nullity}(A) = 2}$

$\boxed{E9}$ $$A = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 6 & 7 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 2 & 6 & 6 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{rref}(A)$$

(1.) $\boxed{\left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}}$ is basis for $Col(A)$, rank $(A) = 2$

(2.) $x \in Null(A) \Rightarrow \begin{aligned} x_1 &= -3x_2 \\ x_3 &= 0 \end{aligned}$ (I read these off rref(A) and a short calculation)

$$\Rightarrow x = (x_1, x_2, x_3) = (-3x_2, x_2, 0) = x_2(-3, 1, 0)$$

$$\Rightarrow Null(A) = \text{span} \underbrace{\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}}_{\text{BASIS FOR } Null(A)}, \quad \nu(A) = 1$$

(Remark: $Ax = 0 \Rightarrow row_i(A) \cdot x = 0 \Rightarrow x \perp row_i(A)$ for $i = 1, 2, .., m$
We must see $Null(A)$ perpendicular to rows of A)

Th^m/ Let $A \in \mathbb{R}^{m \times n}$ have k-pivot columns then A has $(n-k)$-nonpivot columns and

(1.) pivot columns form basis for $Col(A)$

(2.) if $x_{i_1}, x_{i_2}, \dots, x_{i_{n-k}}$ are the nonpivotal variables then there exist LI $v_1, v_2, \dots, v_{n-k}$ for which $Null(A) = span\{v_1, v_2, \dots, v_{n-k}\}$ and in particular $Ax = 0$ has solution

$$x = \sum_{j=1}^{n-k} x_{i_j} v_j$$

In fact, $v_1, v_2, \dots, v_{n-k}$ are derived by rewriting the solution to $Ax = 0$ as a linear combination with weights or coefficients given by $x_{i_1}, x_{i_2}, \dots, x_{i_{n-k}}$

(3.) $rank(A) + nullity(A) = n$

I'll give a simple example which uses the notation above.

E10

$$A = \begin{array}{c} \left[\begin{array}{ccccc} 1 & -2 & 0 & 0 & 0 \\ 2 & -4 & 0 & 0 & 0 \\ 3 & \underbrace{-6 \quad 0}_{\text{nonpivotal columns}} & & 0 & 0 \end{array}\right] \\ \uparrow \text{ pivot column} \end{array} \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underbrace{i_1 = 2, \ i_2 = 3, \ i_3 = 4, \ i_4 = 5}_{\text{enumeration of nonpivotals}}$

$$Ax = 0 \Rightarrow x = (2x_2, x_2, x_3, x_4, x_5)$$

$$= x_2(2,1,0,0,0) + x_3(0,0,1,0,0) + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \sum_{j=1}^{4} x_{i_j} v_j \quad \text{with } v_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Remark: vectors in the basis for $Null(A)$ must be $\perp$ to rows of A

I hope you understand the method to find the basis for Col(A) and Null(A). I think the Col(A) basis is easy, but the Null(A) basis calculation requires some study, but once you understand E7, E8, E9, E10 you ought to be able to generalize. The null space basis calculation is a major checkpoint in this course, it keeps coming up from here on out so learn it well now. What follows is more novel

E11 $W_1 = \text{span}\{(1,1,0,0), (0,0,1,-1)\}$
can we find A for which $\text{Null}(A) = W_1$?
In other words, can we rewrite W_1 as the solution set $Ax = 0$?

$$x = x_2(1,1,0,0) + x_3(0,0,1,-1)$$
$$= (x_2, x_2, x_3, -x_3)$$

$x_1 = x_2 \qquad x_1 - x_2 = 0$

$x_4 = -x_3 \qquad x_3 + x_4 = 0$