SOLUTIONS TO LECTURE 19 PROBLEMS 73-76

P73 Let G be a group and S the set of subgroups of G.
If $a \in G$ and $H \leq G$ then define $a * H = aHa^{-1}$
Show that $*$ forms a group actions and describe orbits & stabilizers.

We need $a * H = aHa^{-1} \leq G$. We can see $aHa^{-1} = \phi_a(H)$ where $\phi_a : G \to G$ is the inner-automorphism induced from $a \in G$ thus $aHa^{-1} \leq G$ as it is the image of a homomorphism.

Note, $e * H = eHe^{-1} = H$ by properties of left & right cosets.

Next, $a * (b * H) = a(b * H)a^{-1} = a(bHb^{-1})a^{-1} = (ab)H(ab)^{-1}$

$$\therefore \quad \underline{a * (b * H) = (ab) * H}.$$

Thus $*$ forms a group action of G on its subgroups.

$$\mathcal{O}(H) = \{a * H \mid a \in G\} = \{aHa^{-1} \mid a \in G\}.$$

the orbit of H is the set of all subgroups conjugate to H.

$$\begin{aligned} G_H &= \{ g \in G \mid gHg^{-1} = H \} \\ &= \{ g \in G \mid gH = Hg \} \\ &= N_G(H) \end{aligned}$$

$\leftarrow$ normalizer of H in G. This is the largest set in G for which $H \trianglelefteq N_G(H)$. For example, $H \trianglelefteq G \Rightarrow N_G(H) = G$.

$\boxed{P74}$ Let $H \leq G$ where $H \neq G$. Let $S = G/H$ denote the set of left-cosets of H. Define $g * (xH) = (gx)H$ for each $g \in G$ and $xH \in G/H$.

$e * (xH) = (ex)H = xH$ and for $a, b \in G$ and $xH \in G/H$ we likewise calculate,

$$
\begin{aligned}
(ab) * (xH) &= ((ab)x)H \\
&= a(bx)H \\
&= a * ((bx)H) \\
&= a * (b * xH)
\end{aligned}
$$

Thus $*$ forms a group action of G on G/H.

$$\underline{\mathcal{O}(xH) = \{(gx)H \mid g \in G\} = G/H}.$$

this action is <u>transitive</u>.

$$G_{xH} = \{g \in G \mid (gx)H = xH\}$$

the condition $gxH = xH \Rightarrow gxx^{-1} \in H \therefore g \in H$

hence $\underline{G_{xH} = H}$. The <u>fixed subset</u>

$$S^G = \{xH \in G/H \mid \underbrace{gxH = xH}_{\Rightarrow g \in H \neq G} \quad \forall g \in G\}$$

$$\therefore \boxed{S^G = \emptyset}$$

$\boxed{P75}$ Let $H, K \leq G$ and define group action of H on G/K by $h \star (gK) = (hg)K$ for each $h \in H$, $g \in G$. Show $|HK| = \dfrac{|H||K|}{|H \cap K|}$

Consider the orbit of K,

$$\mathcal{O}(K) = \{ h \star K \mid h \in H \}$$
$$= \{ hK \mid h \in H \}$$

If $hK = h'K$ then $h'h^{-1} \in K$ where $h, h' \in H$ is assumed thus $hK = h'K \Rightarrow h(H \cap K) = h'(H \cap K)$.

In principle ~~$|\mathcal{O}(K)| = |H|$~~, but this over counts and as each K-coset has $|H \cap K|$-representatives, we calculate $|\mathcal{O}(K)| = \dfrac{|H|}{|H \cap K|}$. Moreover,

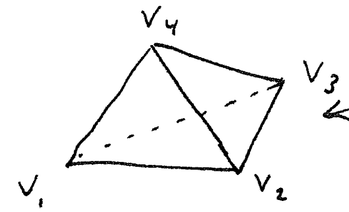
$$G_K = \{ h \in H \mid hK = K \} =$$
$$= \{ h \in H \mid h \in K \}$$
$$= H \cap K$$

Orbit stabilizer yields,

$$|H| = |\mathcal{O}(K)||G_K| = \frac{|H|}{|H \cap K|} |H \cap K| = H$$

GREAT. TRUE, BUT, USELESS. At the moment I don't know how to involve $|HK|$ in this unless I just go back to our action-free sol$^{\underline{n}}$ to this problem (see $\boxed{P60}$)

P76 How many symmetries of a tetrahedron, let us count the ways. Let $G = \Sigma(T)$ symmetries of T. ~ Def^{n} 1.3.10



← T the tetrahedron

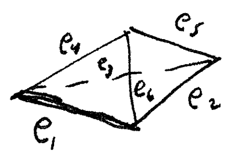
(a.) faces: Let $S = \{f_1, f_2, f_3, f_4\}$ and note that for any face we pick, say f_1, the orbit of the face under G is S. To be precise pick f_1, $\mathcal{O}(f_1) = \{f_1, f_2, f_3, f_4\}$

$$G_{f_1} = \{g \in G \mid g \cdot f_1 = f_1\}$$
$$= \{\sigma : T \to T \mid \sigma(f_1) = f_1 \text{ where } \sigma \in G\}.$$

Geometrically, we see G_{f_1} includes rotations about axis $\perp$ to f_1 through the vertex opposite f_1, we have $0, 2\pi/3, 2(2\pi/3)$ $\therefore$ $|G_{f_1}| = 3$. Hence,

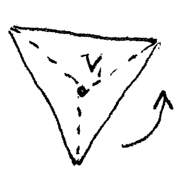
$$|G| = |\mathcal{O}(f_1)||G_{f_1}| = 4(3) = \boxed{12}$$

(b.) Let $S = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. Once more G acts transitively on S and $\mathcal{O}(e_1) = S$.



It can be seen, $G_{e_1} = \{Id, flip\}$ again $|\mathcal{O}(e_1)||G_{e_1}| = (6)(2) = \boxed{12}$.

(c.) Let $S = \{V_1, V_2, V_3, V_4\}$ note $\mathcal{O}(V_1) = S$



$G_{V_1} = \{R_{0°}, R_{120°}, R_{240°}\}$ so, orbit stabilizer Th^{m}

$$|\mathcal{O}(V_1)||G_{V_1}| = (4)(3) = \boxed{12}$$

SOLUTIONS TO LECTURE 20 PROBLEMS 77 – 80

P77 The conjugacy classes in S_5,

representative	# of type
(1)	1
(12)	10
(123)	20
(1234)	30
(12345)	24
(12)(34)	15
(12)(345)	20

Observe, $1 + 10 + 20 + 30 + 24 + 15 + 20 = 120 = 5!$

P78 ~~See Handout on A_n in Course Content.~~ Nope, see over

P79
$$x \equiv 3 \mod 2 \ (m_1)$$
$$x \equiv 4 \mod 7 \ (m_2)$$
$$x \equiv 20 \mod 37 \ (m_3)$$

Solve via Chinese Rem. Th$^{\underline{m}}$ proof.

Identify, $M = 2(7)(37) = \underline{518}$.

$M_1 = 259$, $M_2 = 74$, $M_3 = 14$.

Need to find M_k^{-1} in $\mathbb{Z}_{m_k}$ (these we called y_k in proof)

$\underline{k=1}$ $(259)^{-1}$ in $\mathbb{Z}_2$ well, $259 = 1 \mod 2 \therefore 259^{-1} = 1. = y_1$

$\underline{k=2}$ $(74)^{-1}$ in $\mathbb{Z}_7$ well, $74 = 4 \mod 7 \therefore 74^{-1} = 4^{-1} = 2 = y_2$

$(4 \cdot 2 = 8) \longrightarrow$

$\underline{k=3}$ $(14)^{-1}$ in $\mathbb{Z}_{37}$ requires some calculation, $14^{-1} = 8. = y_3$

Thus
$$x = a_1 M_1 y_1 + a_2 M_2 y_2 + a_3 M_3 y_3$$
$$= 3(259)(1) + 4(74)(2) + 20(14)(8)$$
$$= \boxed{3609}$$

$\boxed{P80}$ Find the conjugacy classes of A_4,

$$A_4 = \{(1), (123), (132), (124), (142), (134), (143), (243), (342), (12)(34), (13)(24), (14)(23)\}$$

to find conjugacy classes we know some things from S_4, we know conjugation preserves cycle-type. However, conjugation may not connect all of a particular type. A class in S_n may split into several in A_n. Let's see how... It is helpful to note,

$$\mathcal{O}(\alpha) = \{\sigma\alpha\sigma^{-1} \mid \sigma \in A_4\} = \text{conjugacy class containing } \alpha.$$

$$G_\alpha = \{\sigma \in A_4 \mid \sigma\alpha\sigma^{-1} = \alpha\} = \text{stabilizer of } \alpha$$

As a check on my calculations, we should find $|\mathcal{O}(\alpha)||G_\alpha| = 12$ for each α.

1.) Note,

$\alpha = (1)$ has $\mathcal{O}(\alpha) = \{\sigma(1)\sigma^{-1} \mid \sigma \in A_4\} = \{(1)\}$.

and $G_{(1)} = \{\sigma \in A_4 \mid \sigma(1)\sigma^{-1} = (1)\} = A_4$. ($1 \cdot 12 = 12$ ✓)

$\therefore \boxed{\mathcal{O}((1)) = \{(1)\}}$

2.) $\alpha = (12)(34)$

p80 continued

2.) $\alpha = (12)(34)$

$(123)(12)(34)(321) = (14)(23) = (14)(23)$

$(321)(12)(34)(123) = (13)(42) = (13)(24)$

It follows $\boxed{\mathcal{O}((12)(34)) = \{(12)(34), (13)(24), (14)(23)\}}$

(we can't reach anything else, cycle-type considerations)

3.) $\alpha = (123)$

$(124)(123)(421) = (1)(243) = (243)$

$(142)(123)(241) = (134)$

$(134)(123)(431) = (1)(243) = (243)$

$(143)(123)(341) = (142)$

$(243)(123)(342) = (142)$

$(342)(123)(243) = (134)$

$(12)(34)(123)(12)(34) = (142)$

$(13)(24)(123)(13)(24) = (134)$

$(14)(23)(123)(14)(23) = (1)(243) = (243)$

Thus, $\boxed{\mathcal{O}((123)) = \{(123), (243), (134), (142)\}}$

4.) $\alpha = (132)$ has $|\mathcal{O}((132))|$ which divides 12 and the orbits partition A_4 thus,

$\boxed{\mathcal{O}((132)) = \{(132), (124), (143), (342)\}}$

In summary, the class eq^{n} is verified as $Z(A_4) = \{1\}$ and,

$$1 + 3 + \underbrace{4 + 4}_{} = 12$$

these we joined in S_4 because the transpositions allowed conjugation of $(123) \longrightarrow (132)$ etc...

SOLUTIONS TO LECTURE 21 : PROBLEMS 81–84

P81 If $A, B \leq G$ and $A \leq N_G(B)$ then $AB \leq G$

$$N_G(B) = \{ g \in G \mid gBg^{-1} = B \}$$

Thus $A \leq N_G(B)$ provides $aBa^{-1} = B \ \forall a \in A$.
or $aB = Ba \ \forall a \in A$ may be useful. Let

$x, y \in AB$ where $x = a_1 b_1$ and $y = a_2 b_2$
and $a_1, a_2 \in A$, $b_1, b_2 \in B$. Consider,

$$\begin{aligned} xy^{-1} &= (a_1 b_1)(a_2 b_2)^{-1} \\ &= a_1 b_1 b_2^{-1} a_2^{-1} \\ &= a_1 a_2^{-1} b_3 \end{aligned}$$

: $b_1 b_2^{-1} \in B$ and $a_2^{-1} \in A$
thus $b_1 b_2^{-1} a_2^{-1} \in B a_2^{-1} = a_2^{-1} B$
$\Rightarrow b_1 b_2^{-1} a_2^{-1} = a_2^{-1} b_3$.

But, $a_1 a_2^{-1} \in A$ as $A \leq G$ and
we've shown $xy^{-1} \in AB$. Noting $e \in A$, $e \in B$
hence $ee = e \in AB \neq \emptyset$ we conclude by
one-step-subgroup test that $AB \leq G$.

$\boxed{P82}$ Suppose $H, K \trianglelefteq G$ and $H \leq K$.

(a.) Prove $K/H \trianglelefteq G/H$.

Since $H \trianglelefteq G$ we have $gHg^{-1} = H \quad \forall g \in G$ hence $gHg^{-1} = H \ \forall g \in K$ as $K \leq G$. We find $H \trianglelefteq K$. Thus K/H is a factor group and we can study its proposed normality w.r.t. G/H. Let $gH \in G/H$ and consider, $xH \in K/H$,

$$\begin{aligned}(gH)(xH)(gH)^{-1} &= (gH)(xH)(g^{-1}H) \qquad (g \in G \text{ and } x \in K) \\ &= (gx)H\, g^{-1}H \\ &= gxg^{-1}H \in K/H \quad \text{since } gxg^{-1} \in K \\ &\qquad\qquad \text{as } gKg^{-1} = K \\ &\qquad\qquad \text{for } K \trianglelefteq G.\end{aligned}$$

Thus, $(gH)(K/H)(gH)^{-1} \subseteq K/H \quad \forall gH \in G/H$
thus $K/H \trianglelefteq G/H$.

(b.) Let $\phi : G/H \longrightarrow G/K$ by $\phi(gH) = gK$ for each $gH \in G/H$. Show ϕ is well-def'd and ϕ homomorphism with $\mathrm{Ker}\,\phi = K/H$.

If $g_1H = g_2H$ then $g_1g_2^{-1} \in H \leq K \therefore g_1g_2^{-1} \in K \Rightarrow g_1K = g_2K$. thus $\phi(g_1) = g_1K = g_2K = \phi(g_2)$. Thus ϕ well-defined.

Also, $\phi(g_1H\,g_2H) = \phi(g_1g_2H) = g_1g_2K = (g_1K)(g_2K) = \phi(g_1H)\phi(g_2H)$.
thus ϕ is homomorphism. Moreover,

$$\begin{aligned}\mathrm{Ker}\,\phi &= \{xH \mid \phi(xH) = K\} \\ &= \{xH \in G/H \mid xK = K\} \\ &= \{xH \in G/H \mid x \in K\} = \underline{K/H = \mathrm{Ker}\,\phi}.\end{aligned}$$

P83

(c.) We have homomorphism $\phi : G/H \longrightarrow G/K$ with $\underline{\text{Ker}\,\phi = K/H \trianglelefteq G/H}$. Moreover, if $xK \in G/K$ then $\phi(xH) = xK$ thus ϕ is a surjection; $\phi(G/H) = G/K$.

Finally, apply the 1^{st} isomorphism Thm,

$$\frac{(G/H)}{\text{Ker}\,\phi} \approx \text{Im}(\phi) = \phi(G/H) = \frac{G}{K}$$

$$\therefore \boxed{\frac{G/H}{K/H} \approx \frac{G}{K}}.$$

Of course, $\text{Ker}\,\phi \trianglelefteq G/H$ so $\underline{K/H \trianglelefteq G/H}$.

$\boxed{P83}$ $G = \mathbb{Z}_{12}$ find subgroup lattice,

$$\begin{tikzcd}[column sep=small]
 & & \langle 1 \rangle = \mathbb{Z}_{12} \arrow[dl, dash] \arrow[d, dash] \\
 & \langle 2 \rangle \arrow[dl, dash] \arrow[d, dash] & \langle 3 \rangle \arrow[dl, dash] \\
\langle 4 \rangle \arrow[dr, dash] & \langle 6 \rangle \arrow[d, dash] & \\
 & \langle 12 \rangle = \{0\} &
\end{tikzcd}$$

Every subgroup is normal so we have many factor groups to consider,

$\mathbb{Z}_{12}/\langle 2 \rangle$	$\mathbb{Z}_{12}/\langle 3 \rangle$	$\mathbb{Z}_{12}/\langle 6 \rangle$	$\mathbb{Z}_{12}/\langle 4 \rangle$
..	..	..	..
$\begin{tikzcd} \langle 1 \rangle \arrow[d, dash] \\ \langle 2 \rangle \end{tikzcd}$	$\begin{tikzcd} \langle 1 \rangle \arrow[d, dash] \\ \langle 3 \rangle \end{tikzcd}$	$\begin{tikzcd}[column sep=tiny] & \langle 1 \rangle \arrow[dl, dash] \arrow[dr, dash] & \\ \langle 2 \rangle \arrow[dr, dash] & & \langle 3 \rangle \arrow[dl, dash] \\ & \langle 6 \rangle & \end{tikzcd}$	$\begin{tikzcd} \langle 1 \rangle \arrow[d, dash] \\ \langle 2 \rangle \arrow[d, dash] \\ \langle 4 \rangle \end{tikzcd}$
$\approx \mathbb{Z}_2$	$\approx \mathbb{Z}_3$	$(\approx \mathbb{Z}_6)$	(this quotient isomorphic to $\mathbb{Z}_4$)

No surprise, remember $\boxed{P58}$, if $m \mid n$ then

$$\mathbb{Z}_n / m\mathbb{Z}_n \approx \mathbb{Z}_m$$

P84 Gallian #51 from pg. 208

Let $\mathbb{Z}[x]$ be polynomials with $\mathbb{Z}$-coeff. under the group operation of addition of polynomials. Show $\psi : \mathbb{Z}[x] \longrightarrow \mathbb{Z}$ defined by

$$\psi(f(x)) = f(3)$$

is a homomorphism. Find Ker ψ, describe, generalize.

Consider, $f(x), g(x) \in \mathbb{Z}[x]$ then

$$\psi(f(x) + g(x)) = f(3) + g(3)$$
$$= \psi(f(x)) + \psi(g(x))$$

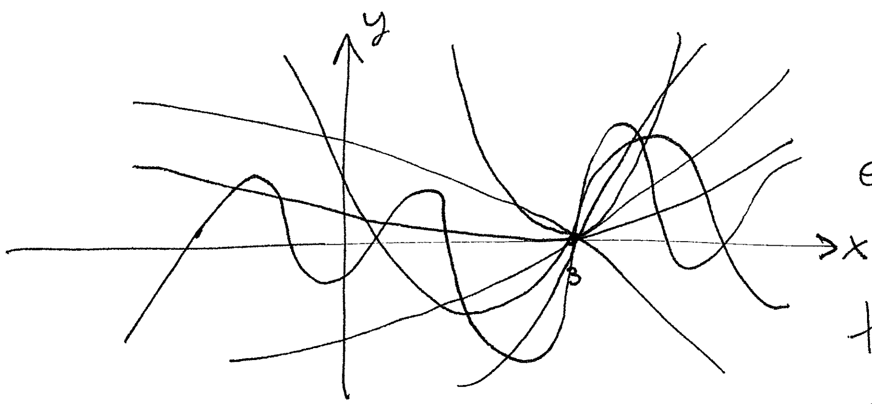
thus ψ is homomorphism.

$$\text{Ker}\,\psi = \{ f(x) \mid f(3) = 0 \}$$

$\Rightarrow f(x) \in \text{Ker}\,\psi$ has $\underbrace{f(x) = (x-3)\,g(x)}_{\text{factor Th}^{\underline{m}}}$ for $g(x) \in \mathbb{Z}[x]$

Ker ψ is polynomial multiples of $(x-3)$.

Geometrically,



etc...

they all go through $(3,0)$.