Lecture 2: Group Actions & Orbit Stabilizer Th^{m}

Group actions put groups to work.

Def^{n}/ Let G be a group with identity e and suppose S is a nonempty set. A <u>multiplication</u> of G on S defined by $*: G \times S \longrightarrow S$ is called a <u>group action</u> of G on S if for each $x \in S$,

(i.) $e * x = x$

(ii.) $a * (b * x) = (ab) * x$ for all $a, b \in G$

$\boxed{E1}$ Consider $S = \mathbb{F}^n$ and $G = Gl(n, \mathbb{F})$, the set of all invertible matrices in $\mathbb{F}^{n \times n}$ where $\mathbb{F}$ is a field (think $\mathbb{F} = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ for now)

$$A * X = AX \qquad \text{(matrix-column multiplication)}$$

gives a group action since

$$I * X = IX = X \quad \text{for} \quad I_{ij} = \delta_{ij}$$

$$A * (B * X) = A * (BX) = A(BX) = (AB)X = (AB) * X.$$

$\boxed{E2}$ If $H \leq Gl(n, \mathbb{F})$ then $A * X = AX$ $\forall A \in H$ and $X \in \mathbb{F}^n$ once more defines an action of H on $\mathbb{F}^n$.

Th^{m}/ If $*: G \times S \longrightarrow S$ defines a group action of G on S then if $H \leq G$ we find $*$ suitably restricted to $\tilde{*}: H \times S \longrightarrow S$ by $h \tilde{*} x = h * x$ $\forall h \in H$ and $x \in S$ defines a group action of H on S.

Let's expand our view of the symmetric group

Def^n/ Let X be a set then $S_X = \{\sigma : X \to X \mid \sigma \text{ bijection}\}$ is the symmetric group of X

Often we've considered $X = \{1, 2, \dots, n\}$ then $S_X = S_n$.

Th^m/ Every group G is isomorphic to subgroup of S_G (CAYLEY)

Proof: for $g \in G$ define $L_g : G \to G$ by $L_g(x) = gx \ \forall x \in G$. Observe $(L_a L_b)(x) = L_a(L_b(x)) = L_a(bx) = a(bx) = (ab)x = L_{ab}(x)$ for all $x \in G$ thus $L_a L_b = L_{ab}$. Moreover, for each $g \in G$ observe $L_{g^{-1}} L_g = L_{g^{-1}g} = L_e = Id_G \therefore L_g \in S_G$. Let $\varphi : G \to S_G$ be defined by $\varphi(g) = L_g$ then note $\varphi(ab) = L_{ab} = L_a L_b = \varphi(a)\varphi(b)$ hence φ homomorphism. If $g \in \mathrm{Ker}(\varphi)$ then $\varphi(g) = Id_G \Rightarrow L_g = Id$ thus $L_g(e) = Id(e) \Rightarrow ge = g = e \therefore \underline{\mathrm{Ker}\,\varphi = \{e\}}$. Consequently, $\varphi(G) \cong G$ since $G/\mathrm{Ker}\,\varphi \cong \varphi(G)$. //

E3 $G = \{1, a, b, c\}$ where $|a| = |b| = |c| = 2$ and $ab = c$ is the Klein 4-group. Let's write $\varphi(G) \leq S_G$ via generalized cycle notation

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

$L_1 = (1)(a)(b)(c) = (1)$

$L_a = (1\,a)(b\,c)$

$L_b = (1\,b)(a\,c)$

$L_c = (1\,c)(a\,b)$

Remark: in §1.7, p. 61 of Rotman AMA 2^nd Ed, he mentions the cycles corresponding the element are the row in the Cayley Table. I almost agree.

Thm/ Given a group action of G on a set T denoted by $*: G \times T \longrightarrow T$ if we define $\alpha_g : T \longrightarrow T$ via $\alpha_g(x) = g * x \ \forall x \in T$ then α_g is a bijection and $\varphi : G \longrightarrow S_T$ given by $\varphi(g) = \alpha_g \ \forall g \in G$ defines a <u>homomorphism</u>.

<u>Proof</u>: let $\alpha_g(x) = g * x \quad \forall g \in G$ and $\forall x \in T$.
Observe, for $a, b \in G$ and $x \in T$,

$$\begin{aligned}
(\alpha_a \circ \alpha_b)(x) &= \alpha_a(\alpha_b(x)) && : \text{def}^{\underline{n}} \text{ of composition} \\
&= a * (b * x) && : \text{def}^{\underline{n}} \text{ of } \alpha \\
&= (ab) * x && : \text{axiom of group action} \\
&= (\alpha_{ab})(x)
\end{aligned}$$

Thus $\alpha_a \circ \alpha_b = \alpha_{ab}$ as the calculation above holds $\forall x \in T$.
Observe $\alpha_e(x) = e * x = x \quad \forall x \in T$ if e is <u>identity</u> of G hence $\alpha_e = Id_T$. Notice $\alpha_g \circ \alpha_{g^{-1}} = \alpha_{gg^{-1}} = \alpha_e = Id_T$
Thus $(\alpha_g)^{-1} = \alpha_{g^{-1}}$ and we find α_g is <u>bijection</u> on T.
Finally, if $\varphi(g) = \alpha_g$ then

$$\varphi(ab) = \alpha_{ab} = \alpha_a \circ \alpha_b = \varphi(a) \circ \varphi(b)$$

Thus $\varphi : G \longrightarrow S_T$ is a homomorphism. //

Defn/ If $g * x = x \quad \forall x \in T$ implies $g = e$ (identity) then $* : G \times T \longrightarrow T$ is called a <u>faithful</u> or <u>effective</u> action.

<u>Remark</u>: $\varphi : G \longrightarrow S_T$ given by $\varphi(g) = \alpha_g$ has
$\text{Ker}\,\varphi = \{ g \in G \mid \varphi(g) = \alpha_g = Id_T \} \therefore \ g \in \text{Ker}\,\varphi \iff g * x = x \ \forall x \in T$
It follows $*$ <u>faithful</u> implies φ is <u>injective</u>.

E4 Let G be a group and notice $g * x = gx$ defines a group action of G on itself. Notice $g * x = x \ \forall x \in G$ implies $g = e$ thus $*$ is faithful. The fact that $\varphi(g) = \alpha_g$ defines homomorphism $\varphi : G \longrightarrow S_G$ and $\varphi(G) \leq S_G$ with $G \cong \varphi(G)$ is usually known as <u>Cayley's Th^m</u>

E5 Let G be a group with $\underbrace{\text{subgroup } H}$ then

not necessarily normal, perhaps G/H not a group.

$$G/H = \{aH \mid a \in G\}$$

define $* : G \times (G/H) \longrightarrow (G/H)$ by

$$\boxed{g * (aH) = gaH}$$

Notice $e * (aH) = eaH = aH$ and

$$\left.\begin{aligned} x*(y*aH) &= x*(yaH) \\ &= x(ya)H \\ &= (xy)aH \\ &= (xy)*aH \end{aligned}\right\}$$

So $*$ defines an action of G on G/H

> **Defn/** G a group and S a set and $* : G \times S \longrightarrow S$ a group action. We define
> (i.) the <u>orbit</u> of $x \in S$ is $\mathcal{O}(x) = \{g * x \mid g \in G\}$
> (ii.) the <u>stabilizer</u> of $x \in S$ is $G_x = \{g \in G \mid g * x = x\}$

<u>Remark</u>: $\mathcal{O}(x) \subseteq S$ whereas $G_x \leq G$ (can prove G_x is <u>subgroup</u>, it's called the "isotropy subgroup")

<u>E5 continued</u>/ $\mathcal{O}(aH) = G/H$, $\underbrace{G_{aH} = ?}_{\text{can you answer?}}$

E5 continued

Notice $bH = ba^{-1}aH \Rightarrow (ba^{-1}) * aH = bH$
this is why $\mathcal{O}(aH) = G/H$, we can move aH
to bH for arbitrary $b \in G$ by acting by ba^{-1}.

$$\begin{aligned} G_{aH} &= \{ g \in G \mid g*(aH) = aH \} \\ &= \{ g \in G \mid gaH = aH \} \\ &= \{ g \in G \mid a^{-1}gaH = H \} \\ &= \{ g \in G \mid a^{-1}ga \in H \} \end{aligned}$$

E6 Let G be a group then G acts on itself via conjugation. In particular, for $g, x \in G$

$$\boxed{g * x = gxg^{-1}}$$

Notice $e * x = exe^{-1} = x$ for each $x \in G$ and
$a*(b*x) = a*(bxb^{-1}) = a(bxb^{-1})a^{-1} = (ab)x(ab)^{-1}$
thus $a*(b*x) = (ab)*x \quad \forall a, b, x \in G$.

$$\underbrace{\mathcal{O}(x) = \{ gxg^{-1} \mid g \in G \}}_{\text{conjugacy class}}$$

$$\begin{aligned} G_x &= \{ g \in G \mid gxg^{-1} = x \} \\ &= \{ g \in G \mid gx = xg \} \\ &= C(x) \end{aligned}$$

$\leftarrow$ centralizer of x in G

Def^n/ Given action $*$ of group G on the set S we define $S/G = \{ \mathcal{O}(x) \mid x \in S \}$ the space of orbits

Our next result justifies the usage of "class" in conjugacy class since the orbits of a group action are equivalence classes and S/G provides a partition of S

Th^m/ Given G acts on set S, the orbits of G partition S. Moreover, for $|S| < \infty$, $|S| = \sum_i |\mathcal{O}(x_i)|$ where one x_i is selected for each orbit of the action.

<u>Proof</u>: Let $*: G \times S \longrightarrow S$ define an action of the group G on the set S. Let $x \sim y$ if $x = g * y$ for some $g \in G$. In other words, $x \sim y$ iff $y \in \mathcal{O}(x)$.

(1.) $x = e * x$ thus $x \sim x$ for any $x \in S$.

(2.) suppose $x \sim y$ then $x = g * y$ for some $g \in G$
Hence $g^{-1} * x = (g^{-1}g) * y = e * y = y$ and thus $y \sim x$

(3.) Suppose $x \sim y$ and $y \sim z$ then $\exists g, h \in G$ for which $x = g * y$ and $y = h * z$ then

$$x = g * y = g * (h * z) = (gh) * z$$

thus $x \sim z$.

We have shown $\sim$ is an <u>equivalence relation</u> on S and $\mathcal{O}(x) = [x] = \{ y \in S \mid x \sim y \}$ is its <u>equivalence class</u> which we know <u>partition</u> S. Thus S/G is collection of non-empty, disjoint sets whose union forms S. $|S| = \sum_i |\mathcal{O}(x_i)|$ follows by counting. //

$Th^{\underline{m}}$ / (ORBIT STABILIZER)

Suppose the group G acts on the set S by $*$. Then if $x \in S$ then $\mathcal{O}(x)$ bijectively corresponds to G/G_x. Furthermore, for G and S finite, $\boxed{|G| = |\mathcal{O}(x)||G_x|}$

Proof: Let $* : G \times S \longrightarrow S$ be group action and let $x \in S$. We define $f : \mathcal{O}(x) \longrightarrow G/G_x$ as follows: for $y \in \mathcal{O}(x)$ $\exists g \in G$ with $y = g * x$ and we define $f(y) = gG_x$.

(1.) f is into since $gG_x \in G/G_x$

(2.) f is single-valued: if $y = g * x$ and $y = h * x$ then $h * x = g * x \Rightarrow g^{-1}h * x = x \;\therefore\; g^{-1}h \in G_x$ and we find $gG_x = hG_x$

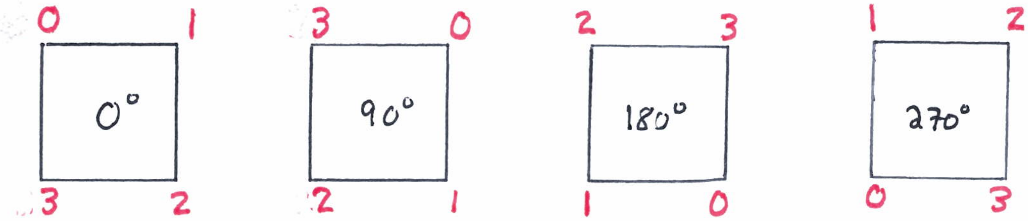
(3.) f is injective: suppose $f(y) = f(z)$ hence $\exists g, h \in G$ for which $y = h * x$ and $z = g * x$ where $hG_x = gG_x$. Observe $h^{-1}g \in G_x \Rightarrow (h^{-1}g) * x = x$ hence $h * x = g * x \Rightarrow \underline{y = z}$.

(4.) f is surjective: suppose $gG_x \in G/G_x$. Notice $g * x = y \in \mathcal{O}(x)$ and $f(y) = gG_x$.

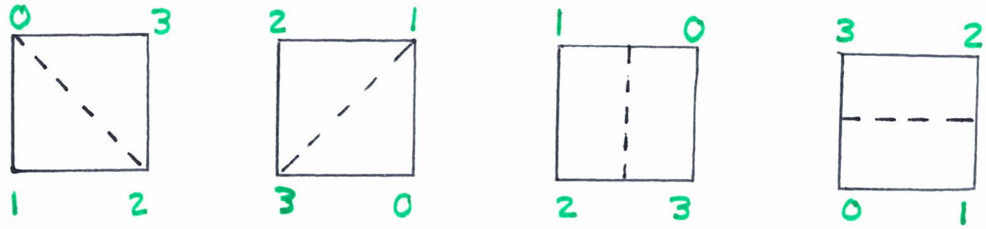
Recall $[G : G_x]$ is the # of G_x-cosets in G. Thus for both S and G finite $f : \mathcal{O}(x) \longrightarrow G/G_x$ a bijection yields $|\mathcal{O}(x)| = |G/G_x| = [G : G_x] = |G|/|G_x|$

$$\therefore \; \underline{|G| = |\mathcal{O}(x)||G_x|} \; .//$$

E7 Consider $S = \{V_0, V_1, V_2, V_3\}$ vertices of square. Abbreviate V_j with j



rotations



reflections

The pictures above illustrate the natural action of D_4 on S.

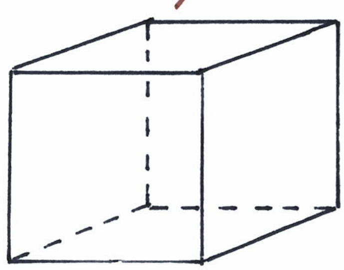
$$\mathcal{O}(V_0) = \{V_0, V_1, V_2, V_3\}$$

$$G_{V_0} = \{(1), (V_1 V_3)\}$$

Observe $|D_4| = 8$ and $|\mathcal{O}(V_0)||G_{V_0}| = 4(2) = 8$.

Def^{n}/ If $* : G \times S \to S$ is group action and $\mathcal{O}(x) = S$ for any $x \in S$ then we say $*$ is <u>transitive</u>

E8 Consider a <u>cube</u> in $\mathbb{R}^3$. Let $G \leq \text{Isom}(\mathbb{R}^3)$ which form symmetries of the cube.



$$\underbrace{S = \{F_F, F_B, F_L, F_R, F_U, F_D\}}_{\text{faces of the cube}}$$

we can rotate any face of the cube to any other face, the action of G on the cube is <u>transitive</u>. The group of isometries which fix a face is D_4 thus

$$|G| = |\mathcal{O}(F_B)||G_{F_B}| = |S||D_4| = 6(8) = \boxed{48}$$

Consider $H \leq G$ with $[G:H] = m$ and let $X = \{aH \mid a \in G\}$ and $\tau_a : X \to X$ by $\tau_a(bH) = abH$ then we can prove $\tau_a \in S_X$ (τ_a a bijection)

Moreover, $\varphi : G \to S_X$ given by $\varphi(a) = \tau_a$ is a group homomorphism. Consider,

$$\begin{aligned}
\text{Ker}\,\varphi &= \{a \in G \mid \varphi(a) = Id_X\} \\
&= \{a \in G \mid \tau_a(x) = x \ \ \forall x \in X\} \\
&= \{a \in G \mid abH = bH \ \ \forall b \in G\} \\
&= \{a \in G \mid b^{-1}ab \in H \ \ \forall b \in G\} \\
&= \{a \in G \mid a \in bHb^{-1} \ \ \forall b \in G\} \\
&= \bigcap_{b \in G} bHb^{-1}
\end{aligned}$$

$$\boxed{\text{Def}^{n}/ \ \text{core}(H) = \{a \in G \mid a \in gHg^{-1} \ \forall g \in G\} = \bigcap_{g \in G} gHg^{-1}}$$

Naturally $\text{core}(H) \trianglelefteq G$ since the kernel of any group homomorphism is normal subgroup.

Lemma: Let $H \leq G$ then

(1.) $\text{core}(H) \trianglelefteq G$ and $\text{core}(H) \subseteq H$

(2.) If $K \trianglelefteq G$ and $K \subseteq H$ then $K \subseteq \text{core}(H)$

Proof: left to reader. //

Th$^{\underline{m}}$/ Extended Cayley Th$^{\underline{m}}$

If $H \leq G$ and $[G:H] = m$ then $\exists\, \theta : G \to S_m$ a group homomorphism with $\text{Ker}\,\theta = \text{core}(H) \subseteq H$.

Proof: Let $X = \{gH \mid g \in G\}$ where $H \leq G$ and $[G:H] = m$. Define $\varphi : G \to S_X$ by $\varphi(g) = \tau_g$ then we have

$$\begin{aligned}\varphi(ab)(gH) &= \tau_{ab}(gH) \\ &= abgH \\ &= a(bgH) \\ &= \tau_a(\tau_b(gH)) \quad \forall gH \in X\end{aligned}$$

thus $\varphi(ab) = \varphi(a) \circ \varphi(b)$ and $\text{Ker}\,\varphi = \text{core}(H)$ and since $[G:H] = m \Rightarrow |X| = m \Rightarrow S_X \cong S_m$ there exists an isomorphism $\psi : S_X \to S_m$ and

$$\psi \circ \varphi : G \xrightarrow{\varphi} S_X \xrightarrow{\psi} S_m$$

consequently $\text{Ker}(\psi \circ \varphi) = \text{Ker}(\varphi) = \text{core}(H)$ $\therefore$ $\theta = \psi \circ \varphi$ serves as homomorphism $\theta : G \to S_m$.

Remark: perhaps you recall if $[G:H] = 2$ then $H \triangleleft G$, well, that generalizes to the following corollary of the Extended Cayley Th$^{\underline{m}}$

Corollary: Let p be the smallest prime dividing the order of a finite group G. Then any subgroup H of G with $[G:H] = p$ is normal

Proof: see Nicholson, 3$^{\text{rd}}$ Ed, pg. 357.

CONJUGACY AND THE CLASS EQUATION

Here I collect some observations which I made in Lecture 17 of Math 421. There are two group actions we look at,

- for $H \leq S_n$ let act on $\mathbb{N}_n = \{1, 2, 3, .., n\}$ via the rule $\sigma \cdot i = \sigma(i)$ for $\sigma \in S_n$, and $i \in \mathbb{N}_n$
- conjugation action of S_n on itself:
$$\beta \cdot \sigma = \beta \sigma \beta^{-1}$$

We should also introduce the fixed-subset

$$\boxed{\text{Def}^{\underline{n}}/ \ S^G = \{x \in S \mid g * x = x, \ \forall g \in G\}}$$

For example, in the context of the conjugacy action,

$$\begin{aligned} S^G &= \{g \in G \mid xgx^{-1} = g \text{ for all } x \in G\} \\ &= \{g \in G \mid gx = xg \ \forall x \in G\} \\ &= Z(G) \end{aligned}$$

$\boxed{E9}$ Consider $S = \{1, 2, 3, 4, 5, 6\}$ and $\sigma = (1, 2, 3)(5, 6)$ then $\langle \sigma \rangle = \{1, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5\} \leq S_6$ then

$$\mathcal{O}(1) = \{1, 2, 3\}$$
$$\mathcal{O}(5) = \{5, 6\}$$
$$\mathcal{O}(4) = \{4\}$$

$$|S| = 6 = \underbrace{|\mathcal{O}(1)|}_{3} + \underbrace{|\mathcal{O}(5)|}_{2} + \underbrace{|\mathcal{O}(4)|}_{1}$$

$\boxed{E10}$ If $\sigma \in S_n$ has orbits $S_1, S_2, .., S_r$ within $\mathbb{N}_n$ then if $|S_i| \geq 2$ for $i = 1, 2, .., k$ then σ can be written in terms of k-disjoint cycles whose entries are found within the orbits $S_1, .., S_k$.

Apparently the orbits of the cyclic group generated by $\sigma \in S_n$ reveal the disjoint cycles of σ.

$\boxed{E11}$ Consider $\alpha = (12)(34)$ in S_4. Let's study the conjugation of α, well a few example calculations at least,

$$(123)\alpha(123)^{-1} = (123)(12)(34)(321) = (14)(23)$$

$$(124)\alpha(124)^{-1} = (124)(12)(34)(421) = (13)(24)$$

$$(23)\alpha(23)^{-1} = (23)(12)(34)(23) = (13)(24)$$

$$(1234)\alpha(1234)^{-1} = (1234)(12)(34)(4321) = (14)(23)$$

$$(12)\alpha(12)^{-1} = (12)(12)(34)(12) = (12)(34)$$

Conjugation preserves cycle-type. Another example calculation, $\beta = (1234)$ and $\gamma = (123)$

$$(12)\beta(12)^{-1} = (12)(1234)(12) = (1342)$$

$$(124)\gamma(124)^{-1} = (124)(123)(421) = (243)$$

Def^{n}/ $\alpha, \beta \in S_n$ have the same cycle-type if they have the same # of each type of cycle in their disjoint cycle factorizations

Orbit Stabilizer Th$^{\underline{m}}$ for conjugacy action

The # of conjugates to x is the index of the centralizer for x. That is, $|\{gxg^{-1} \mid g \in G\}| = [G : C_x]$

Proof: $\mathcal{O}(x) = \{gxg^{-1} \mid g \in G\}$ given $g * x = gxg^{-1}$ and $G_x = \{g \in G \mid gxg^{-1} = x\} = \{g \in G \mid gx = xg\} = C_x$. Now apply the orbit stabilizer Th$^{\underline{m}}$. //

Defⁿ/ The conjugacy class of $x \in G$ is given by $x^G = \{gxg^{-1} \mid g \in G\}$

In our current notational scheme $|x^G| = [G : C(x)]$

Corollary to Orbit Stabilzer for conjugacy action

The # of permutations in S_n with a particular cycle-type must divide $n!$

Proof: since conjugacy classes are formed by permutations of the same cycle type it follows $|\sigma^{S_n}| = \#$ of permutations in S_n with same cycle-type as σ. However, $|\sigma^{S_n}| = [S_n : C(\sigma)]$ and we know $|C(\sigma)|\,[S_n : C(\sigma)] = |S_n| = n!$ $\therefore$ $|\sigma^{S_n}| \,\big|\, n!$.

E12

representative σ	$\lvert\sigma^{S_4}\rvert$ = # of cycles of same type
(1)	1
(12)	6
(123)	8
(1234)	6
$(12)(34)$	3

Observe $1+6+8+6+3 = 24 = 4!$

Defⁿ/ The CLASS EQUATION of a finite group G is

$$|G| = |Z(G)| + \sum_i [G : C(x_i)]$$

where one x_i is selected from each conjugacy class containing more than one element

Th^{m}/ CAUCHY'S Th^{m}: If G is finite group with order divisible by a prime p then G contains an element of order p

Proof: assume G is finite group whose order is divisible by a prime p. We prove the Th^{m} by induction on $|G|$. Observe $|G| = 1$ then the Th^{m} is vacuously true since 1 has no prime divisors. Suppose Th^{m} holds for groups up to order $n-1$ and consider $|G| = n$. If $x \in G$ then $|x^G| = [G : C(x)]$ where $C(x)$ is the centralizer of x and x^G is the conj. class of x.

① If $x \notin Z(G)$ then $|x^G| > 1$ thus $|C(x)| < |G|$. If $p \mid |C(x)|$ for some $x \notin Z(G)$ then by the inductive hypothesis $C(x) \leq G$ has an element of order p and thus G has an elem. of order p.

② If $p \nmid |C(x)|$ for all $x \in G - Z(G)$. Recall we assume $p \mid |G|$ and note

$$|G| = [G : C(x)]\,|C(x)|$$

thus, by Euclid's Lemma, since $p \nmid C(x)$ it must be $p \mid [G : C(x)]$. Since conjugacy classes partition G,

$$|G| = |Z(G)| + \sum_i [G : C(x_i)]$$

Now $p \mid |G|$ and $p \mid [G : C(x_i)]$ for $x_i \in G - Z(G)$ thus $p \mid |Z(G)|$. But $Z(G)$ is abelian and we've shown (in Math 421) that $p \mid Z(G) \Rightarrow Z(G)$ has elem. of order p $\therefore$ G contains element of order p. //