Lecture 30: Tensor Algebras, Symmetric & Exterior Algebras

This Lecture is mostly about §11.5, but we should take a look at §11.2, (this is where $\otimes$ of Chapter 10 is applied to nice case of vector space)

Prop 15 (§11.2)

Let F be subfield of K. If W is m-dim'l vector space over F with basis $w_1, w_2, \ldots, w_m$, then $K \otimes_F W$ is an m-dim'l vector space over K with basis $1 \otimes w_1, \ldots, 1 \otimes w_m$

Consider V, W, X, Y finite dim'l vector spaces over F and $\varphi : V \to X$ and $\psi : W \to Y$ linear transformations then we can construct the tensor product of φ and ψ as follows

$$\varphi \otimes \psi : V \otimes W \longrightarrow X \otimes Y$$
$$(\varphi \otimes \psi)(v \otimes w) = \varphi(v) \otimes \psi(w)$$

Naturally we'd like to understand how to find the matrix of $\varphi \otimes \psi$ given bases for $V \otimes W$ and $X \otimes Y$. Using properties of $\otimes$,

$$\begin{aligned}(\varphi \circ \psi)(v_i \otimes w_j) &= \varphi(v_i) \otimes \psi(w_j) \\ &= \Big(\sum_i \alpha_{pi} x_p\Big) \otimes \Big(\sum_j \beta_{qj} y_q\Big) \\ &= \sum_{i,j} \alpha_{pi} \beta_{qj} (x_p \otimes y_q)\end{aligned}$$

Here $\{x_p\}$ is basis for X and $\{y_q\}$ basis for Y,

$\mathcal{B}_1 = \{v_1, \ldots, v_n\}$ for V

$\mathcal{B}_2 = \{w_1, \ldots, w_m\}$ for W

$\mathcal{E}_1 = \{x_1, \ldots, x_r\}$ for X

$\mathcal{E}_2 = \{y_1, \ldots, y_s\}$ for Y

Then construct bases for $V \otimes W$ and $X \otimes Y$,

$$\mathcal{B} = \{v_i \otimes w_j\} = \{v_1 \otimes w_1, \ldots, v_n \otimes w_m\}$$

$$\mathcal{E} = \{x_i \otimes y_j\} = \{x_1 \otimes y_1, x_1 \otimes y_2, \ldots, x_r \otimes y_1, \ldots, x_r \otimes y_s\}$$

I'd call these lexographically ordered.

- $M_\beta^\varepsilon(\varphi \otimes \psi)$ is an $(r \times n)$-block matrix whose p, q block matrix is the $s \times m$ matrix
$$\alpha_{p,q} M_{\beta_2}^{\varepsilon_2}(\psi)$$

- the matrix for $\varphi \otimes \psi$ is obtained by taking matrix for φ and multiplying each entry by the matrix for ψ.

$\boxed{E1}$ $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(\begin{array}{c|c} 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \hline 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}\right) = \left[\begin{array}{cc|cc} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ \hline 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{array}\right]$$

Def^n/ Let $A = (\alpha_{ij})$ and B be $(r \times n)$ and $(s \times m)$ matrices, respectively, with coefficients from any commutative ring. Then the <u>Kronecker product</u> or <u>tensor product</u> of A and B is the $(rs) \times (nm)$ matrix consisting of $(r \times n)$-block matrix whose (i,j)-th block is the $(s \times m)$-matrix $\alpha_{ij} B$

$\boxed{E2}$

$$R_2(\alpha + i\beta) = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \qquad \lambda = \alpha + i\beta$$

$$N_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_4(\alpha + i\beta) = \left[\begin{array}{c|c} R_2(\lambda) & I_2 \\ \hline 0 & R_2(\lambda) \end{array}\right]$$

$$= \left[\begin{array}{c|c} R_2(\lambda) & 0 \\ \hline 0 & R_2(\lambda) \end{array}\right] + \left[\begin{array}{c|c} 0 & I_2 \\ \hline 0 & 0 \end{array}\right]$$

$$= I_2 \otimes R_2(\lambda) + N_2 \otimes I_2$$

$$R_6(\alpha + i\beta) = I_3 \otimes R_2(\lambda) + N_3 \otimes I_2 = \left[\begin{array}{cc|cc|cc} \alpha & \beta & 1 & 0 & 0 & 0 \\ -\beta & \alpha & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \alpha & \beta & 1 & 0 \\ 0 & 0 & -\beta & \alpha & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & 0 & -\beta & \alpha \end{array}\right]$$

Tensor Algebras, Symmetric & Exterior Algebras

At the end of §10.4 we learn that given R-algebras we can form a new R-algebra; $\mathcal{A}$ and $\mathcal{B}$ R-algebras gives rise to $\mathcal{A} \otimes_R \mathcal{B}$ an R-algebra, the multiplication

$$\boxed{(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)}$$

naturally extends multiplications of $\mathcal{A}$ and $\mathcal{B}$ to $\mathcal{A} \otimes \mathcal{B}$.

Another interesting identity for a commutative ring

$$\boxed{R^s \otimes_R R^t \cong R^{st}}$$

What follows is a bit more general. (§11.5)

Begin with module M over a commutative ring R then we introduce k-tensors of M,

$$T^0(M) = R$$
$$T^1(M) = M$$
$$T^2(M) = M \otimes_R M$$
$$\vdots$$
$$T^k(M) = \underbrace{M \otimes_R M \otimes_R \cdots \otimes_R M}_{k\text{-fold}}$$

$$\boxed{\text{Def}^{\underline{n}}/\ T(M) = R \oplus T^1(M) \oplus T^2(M) \oplus \cdots = \bigoplus_{k=0}^{\infty} T^k(M)}$$

(tensor algebra of M)

Th^{m} (31) If M is any R-module over commutative ring R,

(1.) $T(M)$ is an R-algebra containing M with multiplication defined by mapping

$$(m_1 \otimes \cdots \otimes m_i)(m_1' \otimes \cdots \otimes m_j') = m_1 \otimes \cdots \otimes m_i \otimes m_1' \otimes \cdots \otimes m_j'$$

extended linearly over R. Observe,

$$T^i(M)\, T^j(M) \subseteq T^{i+j}(M)$$

T(M) is graded ring

(2.) $T(M)$ has universal property, if A is any R-algebra and $\varphi : M \longrightarrow A$ is R-mod. homomorphism then $\exists !$ homomorphism $\Phi : T(M) \longrightarrow A$ s.t. $\Phi|_M = \varphi$.

Proposition 32 If $V = \text{span}_F(\beta)$ where F is field and $\beta = \{v_1, v_2, \ldots, v_n\}$ then the k-tensors

$$v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$$

where $v_{i_j} \in \beta$ form a vector space basis for $T^k(V)$ over F hence $\dim(T^k(V)) = n^k$

E3 $R = \mathbb{Z}$, $M = \mathbb{Q}/\mathbb{Z}$

we could show $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) = 0$

thus $T(M) = T^0(M) \oplus T^1(M)$

$$\underline{T(\mathbb{Q}/\mathbb{Z}) = \mathbb{Z} \oplus (\mathbb{Q}/\mathbb{Z})}$$

E4 $(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$

$$\Rightarrow T(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_n \oplus \mathbb{Z}_n \oplus \cdots \cong \frac{\mathbb{Z}[x]}{(nx)}.$$

Def^n/ ① A ring S is called a <u>graded ring</u> If $S = S_0 \oplus S_1 \oplus S_2 \oplus \cdots$ such that $S_i S_j \subseteq S_{i+j}$ we say element of S_k are <u>homogeneous of degree k</u>

② An ideal I of graded ring S is called a <u>graded ideal</u> if $I = \bigoplus_{k=0}^{\infty} (I \cap S_k)$

③ A ring homomorphism $\varphi : S \longrightarrow T$ between two graded rings S and T is a homomorphism of graded rings if $\varphi(S_k) \subseteq T_k$ for $k = 0, 1, 2, \ldots$

<u>Prop. 33</u> | S/I is graded ring with homogeneous component isomorphic to S_k / I_k (with S and I as above)

And now for the main events,

Def^n/ The <u>symmetric algebra</u> of an R-module M is the R-algebra formed by the construction of a commutative quotient ring of $T(M)$. In particular, we quotient $T(M)$ by the ideal $C(M)$ generated by all elements of the form $m_1 \otimes m_2 - m_2 \otimes m_1$ for all $m_1, m_2 \in M$. We denote

$$S(M) = T(M) / C(M)$$

Th$^{\underline{m}}$ (34) Let M be an R-module over commutative ring R and let $S(M)$ be its symmetric algebra.

(1.) the $k^{\underline{th}}$ symmetric power $S^k(M)$ can be obtained by the quotient of $\underbrace{M \otimes M \otimes \cdots \otimes M}_{k\text{-fold}}$ by all element of the form $m_1 \otimes \cdots \otimes m_k - m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(k)}$ for all $m_i \in M$ and permutations $\sigma \in S_k$.

(2.) If $\varphi : M \times \cdots \times M \to N$ is symmetric k-multilinear map over R then $\exists!$ R-mod. homomorphism $\Phi : S^k(M) \to N$ s.t. $\Phi =$ $\varphi = \Phi \circ \iota$ where $\iota(m_1, \dots, m_k) = m_1 \otimes \cdots \otimes m_k \mod C(M)$.

(3.) If A is any commutative R-algebra and $\varphi : M \to A$ is R-module homomorphism, then $\exists!$ R-algebra homomorphism $\Phi : S(M) \to A$ where $\Phi|_M = \varphi$.

Corollary (35) V a finite dim'l vector space of dimension n over F. Then $S(V)$ is isomorphic as a graded F-algebra to the ring of polynomials in n-variables over F. That is $\exists$ an isomorphism of vector spaces from $S^k(V)$ onto space of homogeneous polynomials of degree k $\Rightarrow$ $\dim(S^k(V)) = \binom{k+n-1}{n-1}$.

Remark: $v \odot w = v \otimes w + w \otimes v$ is the symetrized tensor product, the analog of $v \wedge w = v \otimes w - w \otimes v$.

That said, $\wedge$ is defined a bit less concretely in what follows $m_1 \wedge m_2 \wedge \cdots \wedge m_k = m_1 \otimes m_2 \otimes \cdots \otimes m_k \mod \Lambda(M)$

$\Lambda(M)$ generated by $m \otimes m$ so $\underbrace{(m+m') \wedge (m+m')}_{} = 0 \mod \Lambda(M)$

$$m \wedge m + m \wedge m' + m' \wedge m + m' \wedge m' = 0$$

$$\Rightarrow \quad m \wedge m' = -m' \wedge m \mod \Lambda(M)$$

Def^n/ The <u>exterior algebra</u> of an R-module M is the R-module formed by $T(M)/A(M)$ where $A(M)$ is the ideal generated by all elements of the form $m \otimes m$ for $m \in M$. The <u>exterior algebra</u> $T(M)/A(M) = \Lambda(M)$ and the image of $m_1 \otimes m_2 \otimes \cdots \otimes m_k$ in $\Lambda(M)$ is denoted $m_1 \wedge m_2 \wedge \cdots \wedge m_k$.

Th^m (36) Let M be an R-module over commutative ring R and let $\Lambda(M)$ be its exterior algebra.

(1.) the k^{th} exterior product $\Lambda^k(M)$ is given by $\underbrace{M \otimes M \otimes \cdots \otimes M}_{k\text{-fold}}$ modulo submodule generated by elements of the form $m_1 \otimes m_2 \otimes \cdots \otimes m_k$ where $m_i = m_j$ for some $i \neq j$.

(2.) If $\varphi : M \times \cdots \times M \longrightarrow N$ is an alternating k-multilinear map then $\exists!$ R-module homomorphism $\Phi : \Lambda^k(M) \longrightarrow N$ such that $\varphi = \Phi \circ \iota$ and $\iota : M \times M \times \cdots \times M$ is defined by $\iota(m_1, \dots, m_k) = m_1 \wedge \cdots \wedge m_k$

<u>Proof</u>: the k-tensors in $A^k(M)$ in the ideal $A(M)$ generated by $m \otimes m$ have the form

$$m_1 \otimes \cdots \otimes m_{i-1} \otimes (m \otimes m) \otimes m_{i+2} \otimes \cdots \otimes m_k$$

which is the form described in (1.) hence $S_{\text{generated by (1)}} \supseteq \Lambda^k(M)$

Proof continued

Consider,

$$m' \otimes m = -m \otimes m' + [(m+m') \otimes (m+m') - m \otimes m - m' \otimes m']$$
$$= -m \otimes m' \mod \mathcal{A}(M)$$

Hence interchanging any two consecutive entries $\otimes$'d in a simple k-tensor results in a $(-1) \mod \mathcal{A}(M)$

Then for $\underbrace{m_1 \otimes m_2 \otimes \cdots \otimes m_i \otimes \cdots \otimes m_j \otimes \cdots \otimes m_k}$ where $i \neq j$ yet $m_i = m_j$ $\quad \mathcal{M} = \pm\, m_1 \otimes \cdots \otimes m_i \otimes m_j \otimes \cdots \otimes m_k \in \mathcal{A}^k(M)$

Thus $(\,\$_{\text{generated by (1)}} \subseteq \Lambda^k(M)$ and (1.) follows.

$\boxed{E5}$ Suppose $V = \text{span}\{v\}$ over F then $v \wedge v = 0$ thus $\Lambda^k V = 0$ for $k \geq 2$ and $\underline{\Lambda(V) = F \oplus V \oplus 0 \oplus 0 \oplus \cdots}$ $(\dim(\Lambda(V)) = 2^1 = 2.)$

$\boxed{E6}$ $V = \text{span}\{a, b\}$ then $a \wedge b$ is the only nontrivial 2-vector over V and $\Lambda^k V = 0$ for $k \geq 3$

$$\underline{\Lambda(V) = F \oplus V \oplus F(a \wedge b) \oplus 0 \oplus \cdots} \quad (\dim(\Lambda(V)) = 2^2 = 4)$$

Remark: see p. 448 of D&F for how to derive my assertions here from quotient by $\underbrace{\mathcal{A}(M)}_{\text{generated by } m \otimes m}$ of $\underbrace{\mathcal{T}(M)}_{\text{tensor algebra}}$

$\boxed{E7}$ $V = \text{span}_F\{dx, dy, dz\}$

$dx \wedge dy,\ dy \wedge dz,\ dz \wedge dx \in \Lambda^2(V)$

$dx \wedge dy \wedge dz \in \Lambda^3(V)$

$$\Lambda(V) = F \oplus V \oplus F(dy \wedge dz, dz \wedge dx, dx \wedge dy) \oplus F(dx \wedge dy \wedge dz) \oplus 0 \oplus 0 \oplus \cdots$$

$$\dim(\Lambda(V)) = 1 + 3 + 3 + 1 = 8 = 2^3.$$

Corollary (37)

Let V be finite dim'l over F with basis $\beta = \{v_1, v_2, \ldots, v_n\}$ then the vectors

$$v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} \quad \text{for } 1 \leq i_1 < i_2 < \cdots < i_k \leq n$$

give a basis for $\Lambda^k(V)$ for $k \leq n$ and

$\Lambda^k(V) = 0$ for $k \geq n+1$. Note $\dim(\Lambda^k(V)) = \binom{n}{k}$

Remark: proof on p. 449 of D&F, it is mentioned the result above applies to free R-module of rank n.