# Lecture 32: Smith Normal Form and Modules over PID

If $R$ is a PID and $M$ is a finitely generated $R$-module then we can calculate the Smith Normal Form of an appropriate matrix and it will reveal the invariant factors for the decomposition of $M \cong R^r \oplus \mathrm{Tor}(M)$. Let's review the Fundamental Th$^{m}$ Invariant Factor Form

**Th$^{m}$/** Let $R$ be a PID and $M$ a finitely generated $R$-module,

(1.) $M \cong R^r \oplus R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_m)$ for some integer $r \geq 0$ and nonzero, non-units $\underbrace{a_1, a_2, .., a_m}_{} \in R$ for which $a_1 \mid a_2 \mid \cdots \mid a_m$ — the invariant factors

(2.) $M$ is torsion free iff $M$ is free.

(3.) In the decomposition given in (1.),

$$\mathrm{Tor}(M) \cong R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_m)$$

In particular, $M$ is torsion module iff $r = 0$ and in this case $\mathrm{ann}(M) = (a_m)$

When $R$ is a PID it is also a UFD hence the invariant factors can be decomposed into prime powers and this leads us to:

**Th$^{m}$/ Fundamental Th$^{m}$ (Elementary Divisor Form)**

Given $R$ a PID and $M$ a finitely generated $R$-module then $M$ is direct sum of cyclic modules whose annihilators are either $(0)$ or generated by powers of primes in $R$,

$$M \cong R^r \oplus R/(p_1^{\alpha_1}) \oplus R/(p_2^{\alpha_2}) \oplus \cdots \oplus R/(p_t^{\alpha_t})$$

where $r \geq 0$ is an integer and $\boxed{p_1^{\alpha_1}, .., p_t^{\alpha_t}}$ are positive powers of not necessarily distinct primes $p_1, .., p_t$

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**Def^n/** The elementary divisors of $M$ are $p_1^{\alpha_1}, p_2^{\alpha_2}, \dots, p_t^{\alpha_t}$ when $M \cong R^r \oplus R/(p_1^{\alpha_1}) \oplus R/(p_2^{\alpha_2}) \oplus \cdots \oplus R/(p_t^{\alpha_t})$

We can collect like primes together and form the Primary Decomposition Th^m,

**Th^m (7)** THE PRIMARY DECOMPOSITION THEOREM

Let $R$ be a P.I.D. and let $M$ be a nonzero torsion $R$-module (not necessarily finitely generated) with nonzero annihilator $a$. Suppose $a$ factors into distinct prime powers in $R$ via

$$a = u\, p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$$

and let $N_i = \{ x \in M \mid p_i^{\alpha_i} x = 0 \}$ for $1 \le i \le n$.

Then $N_i$ is submodule of $M$ with annihilator $p_i^{\alpha_i}$ and is the submodule of $M$ of all elements annihilated by some power of $p_i$. We have

$$M = N_1 \oplus N_2 \oplus \cdots \oplus N_n$$

If $M$ is finitely generated then each $N_i$ is the direct sum of finitely many cyclic modules whose annihilators are divisors of $p_i^{\alpha_i}$

**Def^n/** $N_i$ is the $p_i$-primary component of $M$

Remark: the case of $M = V$ which is an $F[x]$-module given by $T : V \to V$ linear and $f(x)V = f(T)V$ has $N_i = K_{\lambda_i} \leftarrow$ generalized eigenspace with eigenvalue $\lambda_i$ (assuming $p_T(x)$ splits over $F$)

Given two $R$-modules $M_1$ and $M_2$ we can determine if $M_1 \cong M_2$ by comparing their free rank $(r_1 = r_2)$ and their list of <u>invariant</u> <u>factors</u> or their list of <u>elementary</u> <u>divisors</u> coincide. (this is Th$^{\underline{m}}$(9) on p. 466 of §12.1 of Dummit and Foote)

Th$^{\underline{m}}$ (21) Let $A$ be an $n \times n$ matrix over the field $F$. Then the $(n \times n)$-matrix $xI - A$ with entries $F[x]$ can be put into diagonal form (by row and column operations over $F[x]$)

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & a_1(x) & & & \\ & & & & a_2(x) & & \\ & & & & & \ddots & \\ & & & & & & a_m(x) \end{pmatrix}$$

with monic nonzero elements $a_1(x), a_2(x), \dots, a_m(x)$ of $F[x]$ with degrees at least one and satisfying

$$a_1(x) \mid a_2(x) \mid \dots \mid a_m(x)$$

There are the <u>invariant</u> <u>factors</u> for the $F[x]$-module given by $T = L_A$ (left mult. by $A$)