Chapter 3

multivariate limits

The first two sections of this chapter will not be covered in lecture in their entirety. However, I would like you to read through it and try to get the general idea. You should get a good sense of what parts of the first two sections will be tested from the discussion in lecture and your problem set. The third section contains the problems that are typically tested on this material in a generic calculus III course.

3.1 open sets

In this section we describe the *euclidean topology* for $\mathbb{R}^n$. In the study of functions of one real variable we often need to refer to open or closed intervals. The definition that follows generalizes those concepts to n-dimensions.

Definition 3.1.1.

An **open ball** of radius ϵ centered at $\vec{a} \in \mathbb{R}^n$ is the subset of all points in $\mathbb{R}^n$ which are less than ϵ units from $\vec{a}$, we denote this open ball by

$$B_\epsilon(\vec{a}) = \{\vec{x} \in \mathbb{R}^n \mid ||\vec{x} - \vec{a}|| < \epsilon\}$$

The **closed ball** of radius ϵ centered at $\vec{a} \in \mathbb{R}^n$ is likewise defined

$$\overline{B}_\epsilon(\vec{a}) = \{\vec{x} \in \mathbb{R}^n \mid ||\vec{x} - \vec{a}|| \leq \epsilon\}$$

Notice that in the $n = 1$ case we observe an open ball is an open interval: let $a \in \mathbb{R}$,

$$B_\epsilon(a) = \{x \in \mathbb{R} \mid ||x - a|| < \epsilon\} = \{x \in \mathbb{R} \mid |x - a| < \epsilon\} = (a - \epsilon, a + \epsilon)$$

In the $n = 2$ case we observe that an open ball is an open disk: let $(a, b) \in \mathbb{R}^2$,

$$B_\epsilon((a, b)) = \big\{(x, y) \in \mathbb{R}^2 \mid ||\,(x, y) - (a, b)\,|| < \epsilon\big\} = \big\{(x, y) \in \mathbb{R}^2 \mid \sqrt{(x - a)^2 + (y - b)^2} < \epsilon\big\}$$

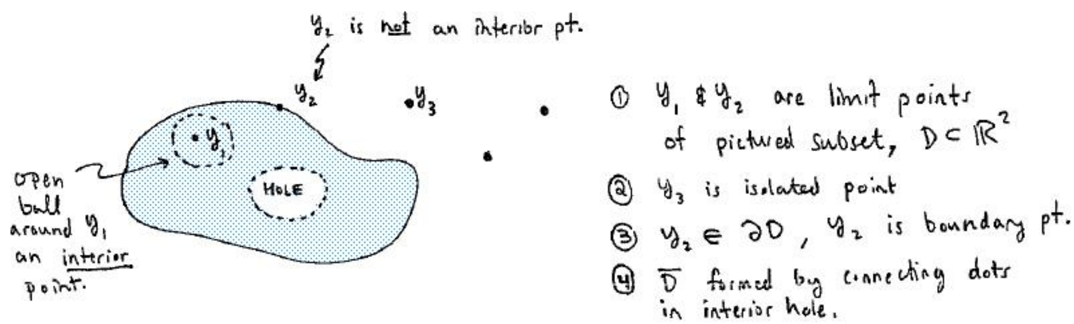
For $n = 3$ an open-ball is a sphere without the outer shell. In contrast, a closed ball in $n = 3$ is a solid sphere which includes the outer shell of the sphere.

$$B_\epsilon((a,b,c)) = \left\{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \epsilon\right\}$$

Definition 3.1.2.

Let $D \subseteq \mathbb{R}^n$. We say $\vec{y} \in D$ is an **interior point** of D iff there exists some open ball centered at $\vec{y}$ which is completely contained in D. We say $\vec{y} \in \mathbb{R}^n$ is a **limit point** of D iff every open ball centered at $\vec{y}$ contains points in $D - \{\vec{y}\}$. We say $\vec{y} \in \mathbb{R}^n$ is a **boundary point**[1] of D iff every open ball centered at $\vec{y}$ contains points not in D and other points which are in $D - \{\vec{y}\}$. We say $\vec{y} \in D$ is an **isolated point** of D if there exist open balls about $\vec{y}$ which do not contain other points in D. The set of all interior points of D is called the **interior of** D. Likewise the set of all boundary points[2] for D is denoted ∂D. The **closure** of D is defined to be $\overline{D} = D \cup \{\vec{y} \in \mathbb{R}^n \mid \vec{y} \text{ a limit point}\}$

If you're like me the paragraph above doesn't help much until I see the picture below. All the terms are aptly named. The term "limit point" is given because those points are the ones for which it is natural to define a limit.



Definition 3.1.3.

Let $A \subseteq \mathbb{R}^n$ is an **open set** iff for each $\vec{x} \in A$ there exists $\epsilon > 0$ such that $\vec{x} \in B_\epsilon(\vec{x})$ and $B_\epsilon(\vec{x}) \subseteq A$. Let $B \subseteq \mathbb{R}^n$ is an **closed set** iff its complement $\mathbb{R}^n - B = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \notin B\}$ is an open set.

Notice that $\mathbb{R} - [a,b] = (\infty, a) \cup (b, \infty)$. It is not hard to prove that open intervals are open hence we find that a closed interval is a closed set. Likewise it is not hard to prove that open balls are open sets and closed balls are closed sets.

3.2 the multivariate limit and continuity

The definition of the limit here is the natural generalization of the $\epsilon\delta$-defn. we studied in before.

Definition 3.2.1.

Let $\vec{f} : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping. We say that $\vec{f}$ has limit $\vec{b} \in \mathbb{R}^m$ at limit point $\vec{a}$ of U iff for each $\epsilon > 0$ there exists a $\delta > 0$ such that $\vec{x} \in \mathbb{R}^n$ with $0 < ||\vec{x} - \vec{a}|| < \delta$ implies $||\vec{f}(\vec{x}) - \vec{b}|| < \epsilon$. In such a case we can denote the above by stating that

$$\lim_{\vec{x} \to \vec{a}} \vec{f}(\,\vec{x}\,) = \vec{b}.$$

In single variable calculus the limit of a function is defined in terms of deleted open intervals centered about the limit point. We just defined the limit of a mapping in terms of deleted open balls centered at the limit point. The term "deleted" refers to the fact that we assume $0 < ||\vec{x} - \vec{a}||$ which means we do not consider $\vec{x} = \vec{a}$ in the limiting process. In other words, the limit of a mapping considers values close to the limit point but not necessarily the limit point itself. The case that the function is defined at the limit point is special, when the limit and the mapping agree then we say the mapping is continuous at that point.

Definition 3.2.2.

Let $\vec{f} : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping. If $\vec{a} \in U$ is a limit point of $\vec{f}$ then we say that $\vec{f}$ is **continuous at** $\vec{a}$ iff

$$\lim_{\vec{x} \to \vec{a}} \vec{f}(\,\vec{x}\,) = \vec{f}(\,\vec{a}\,)$$

If $\vec{a} \in U$ is an isolated point then we also say that $\vec{f}$ is continous at $\vec{a}$. The mapping $\vec{f}$ is **continous on** S iff it is continous at each point in S. The **mapping** $\vec{f}$ **is continuous** iff it is continuous on its domain.

Notice that in the $m = n = 1$ case we recover the definition of continuous functions on $\mathbb{R}$. In practice we seldom calculate a multivariate limit in terms of an $\epsilon - \delta$ argument. I include these to better illustrate just how the definition is directly implemented. In fact, to those of you who are not math majors feel free to skip the proofs in the remainder of this section. I do think it is worthwhile for everyone to at least read the results which are known about continuity of multivariate functions. The section that follows contains problems which I expect you to be able to work yourself by the next test. This section is for breadth and depth of concept.

Example 3.2.3. Claim: the identity function $Id : \mathbb{R}^n \to \mathbb{R}^n$ defined by $Id(\vec{x}) = \vec{x}$ is continuous on $\mathbb{R}^n$.

Proof: *Let $\epsilon > 0$ and choose $\delta = \epsilon$. If $\vec{x} \in \mathbb{R}^n$ such that $0 < ||\vec{x} - \vec{a}|| < \delta$ then it follows that $||\vec{x} - \vec{a}|| < \epsilon.$. Therefore, $\lim_{\vec{x} \to \vec{a}} \vec{x} = \vec{a}$ which means that $\lim_{\vec{x} \to \vec{a}} Id(\vec{x}) = Id(\vec{a})$ for all $\vec{a} \in \mathbb{R}^n$. Hence Id is continuous on $\mathbb{R}^n$ which means Id is continuous.*

Example 3.2.4. Claim: the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(\vec{x}) = ||\vec{x}||^2$ is continuous on $\mathbb{R}^n$. *To prepare for the proof consider we must show that for an appropriately chosen δ the condition $||\vec{x}-\vec{a}|| < \delta$ implies the difference $|f(\vec{x}) - f(\vec{a})| < \epsilon$. Suppose $||\vec{x}-\vec{a}|| < \delta$ and observe:*

$$\begin{aligned}
|f(\vec{x}) - f(\vec{a})| &= |\vec{x}\bullet\vec{x} - \vec{a}\bullet\vec{a}| \\
&= |\vec{x}\bullet\vec{x} - \vec{x}\bullet\vec{a} + \vec{a}\bullet\vec{x} - \vec{a}\bullet\vec{a}| \\
&= |\vec{x}\bullet(\vec{x}-\vec{a}) + \vec{a}\bullet(\vec{x}-\vec{a})| \\
&= |(\vec{x}+\vec{a})\bullet(\vec{x}-\vec{a})| \\
&\leq ||\vec{x}+\vec{a}||\,||\vec{x}-\vec{a}|| \\
&< \delta\,||\vec{x}+\vec{a}|| \\
&= \delta\,||\vec{x}-\vec{a}+2\vec{a}|| \\
&\leq \delta\left[||\vec{x}-\vec{a}|| + 2||\vec{a}||\right] \\
&< \delta\left[\delta + 2||\vec{a}||\right] \qquad \star
\end{aligned}$$

This suggests we choose δ such that $\delta^2 + 2\delta||\vec{a}|| \leq \epsilon$. Let's go for equality, let $||\vec{a}|| = a$ and solve $\delta^2 + 2a\delta - \epsilon = 0$ to find $\delta = -a \pm \sqrt{a^2+\epsilon}$. The solution $\delta = -a + \sqrt{a^2+\epsilon}$ is clearly positive for $\epsilon > 0$.

Proof: *Let $\epsilon > 0$ and let $\vec{a} \in \mathbb{R}^n$ with $||\vec{a}|| = a$ choose $\delta = -a + \sqrt{a^2+\epsilon}$ which is clearly positive. Suppose $\vec{x} \in \mathbb{R}^n$ and $0 < ||\vec{x}-\vec{a}|| < \delta$ and calculate, following the $\star$ calculation,*

$$|f(\vec{x}) - f(\vec{a})| = \delta^2 + 2a\delta = \epsilon.$$

Therefore, by the definition of the limit, $\lim_{\vec{x}\to\vec{a}} ||\vec{x}||^2 = ||\vec{a}||^2$. and since $\vec{a}$ is arbitrary this shows f is continuous on all of $\mathbb{R}^n$. $\square$

The examples that follow are somewhat abstract, but their use is astounding once they're paired with a couple basic theorems about the multivariate limit.

Proposition 3.2.5.

Let $\vec{f} : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping with component functions $f_1, f_2, \dots, f_m$ hence $\vec{f} = (f_1, f_2, \dots, f_m)$. If $\vec{a} \in U$ is a limit point of $\vec{f}$ then

$$\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{b} \qquad \Leftrightarrow \qquad \lim_{\vec{x}\to\vec{a}} f_j(\vec{x}) = b_j \text{ for each } j = 1, 2, \dots, m.$$

.

Proof: $(\Rightarrow)$ Suppose $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{b}$. Then for each $\epsilon > 0$ choose $\delta > 0$ such that $0 < ||\vec{x}-\vec{a}|| < \delta$ implies $||\vec{f}(\vec{x}) - \vec{b}|| < \epsilon$. This choice of δ suffices for our purposes as:

$$|f_j(\vec{x}) - b_j| = \sqrt{(f_j(\vec{x}) - b_j)^2} \leq \sqrt{\sum_{j=1}^{m}(f_j(\vec{x}) - b_j)^2} = ||\vec{f}(\vec{x}) - \vec{b}|| < \epsilon.$$

Hence we have shown that $\lim_{\vec{x}\to\vec{a}} f_j(\vec{x}) = b_j$ for all $j = 1, 2, \dots m$.

($\Leftarrow$) Suppose $\lim_{\vec{x}\to\vec{a}} f_j(\vec{x}) = b_j$ for all $j = 1, 2, \dots m$. Let $\epsilon > 0$. Note that $\epsilon/m > 0$ and therefore by the given limits we can choose $\delta_j > 0$ such that $0 < ||\vec{x} - \vec{a}|| < \delta$ implies $||f_j(\vec{x}) - b_j|| < \sqrt{\epsilon/m}$. Choose $\delta = min\{\delta_1, \delta_2, \dots, \delta_m\}$ clearly $\delta > 0$. Moreoever, notice $0 < ||\vec{x} - \vec{a}|| < \delta \leq \delta_j$ hence requiring $0 < ||\vec{x} - \vec{a}|| < \delta$ automatically induces $0 < ||\vec{x} - \vec{a}|| < \delta_j$ for all j. Suppose that $\vec{x} \in \mathbb{R}^n$ and $0 < ||\vec{x} - \vec{a}|| < \delta$ it follows that

$$||\vec{f}(\vec{x}) - \vec{b}|| = ||\sum_{j=1}^{m}(f_j(\vec{x}) - b_j)e_j|| = \sqrt{\sum_{j=1}^{m}|f_j(\vec{x}) - b_j|^2} \leq \sum_{j=1}^{m}(\sqrt{\epsilon/m})^2 < \sum_{j=1}^{m}\epsilon/m = \epsilon.$$

Therefore, $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{b}$ and the proposition follows. $\square$

We can analyze the limit of a mapping by analyzing the limits of the component functions:

Example 3.2.6. .

Let $f(x) = (\sqrt{x^2}, \sin(x), \frac{\sin x}{x})$ thus $f = (f_1, f_2, f_3)$

where $f_1(x) = \sqrt{x^2}$, $f_2(x) = \sin(x)$, $f_3(x) = \frac{\sin x}{x}$ for $x \in \mathbb{R} - \{0\}$.

$$\left.\begin{array}{l} \lim_{x\to 0} f_1(x) = \sqrt{0^2} = 0 \\ \lim_{x\to 0} (\sin(x)) = 0 \\ \lim_{x\to 0} \left(\frac{\sin x}{x}\right) = 1 \end{array}\right\} \lim_{x\to 0} \left(\sqrt{x^2}, \sin x, \frac{\sin x}{x}\right) = (0, 0, 1).$$

The following follows immediately from the preceding proposition.

Proposition 3.2.7.

Suppose that $\vec{f} : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ is a mapping with component functions $f_1, f_2, \dots, f_m$. Let $\vec{a} \in U$ be a limit point of $\vec{f}$ then $\vec{f}$ is continous at $\vec{a}$ iff f_j is continuous at $\vec{a}$ for $j = 1, 2, \dots, m$. Moreover, $\vec{f}$ is continuous on S iff all the component functions of $\vec{f}$ are continuous on S. Finally, a mapping $\vec{f}$ is continous iff all of its component functions are continuous. .

Proposition 3.2.8.

The cartesian coordinate functions are continuous. The identity mapping is continuous.

Proof: Since the cartesian coordinate functions are component functions of the identity mapping it follows that the coordinate functions are also continuous (using the previous proposition and the fact we showed the identity function was continuous earlier in this chapter). $\square$

Definition 3.2.9.

The **sum** and **product** are functions from $\mathbb{R}^2$ to $\mathbb{R}$ defined by

$$s(x,y) = x + y \qquad p(x,y) = xy$$

Proposition 3.2.10.

The sum and product functions are continuous.

Preparing for the proof: Let the limit point be (a,b). Consider what we wish to show: given a point (x,y) such that $0 < ||(x,y) - (a,b)|| < \delta$ we wish to show that

$$|s(x,y) - (a+b)| < \epsilon \qquad \text{or for the product} \qquad |p(x,y) - (ab)| < \epsilon$$

follow for appropriate choices of δ. Think about the sum for a moment,

$$|s(x,y) - (a+b)| = |x + y - a - b| \leq |x - a| + |y - b|$$

I just used the triangle inequality for the absolute value of real numbers. We see that if we could somehow get control of $|x-a|$ and $|y-b|$ then we'd be getting closer to the prize. We have control of $0 < ||(x,y) - (a,b)|| < \delta$ notice this reduces to

$$||(x-a, y-b)|| < \delta \quad \Rightarrow \quad \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

it is clear that $(x-a)^2 < \delta^2$ since if it was otherwise the inequality above would be violated as adding a nonegative quantity $(y-b)^2$ only increases the radicand resulting in the squareroot to be larger than δ. Hence we may assume $(x-a)^2 < \delta^2$ and since $\delta > 0$ it follows $\boxed{|x-a| < \delta}$. Likewise, $\boxed{|y-b| < \delta}$. Thus

$$|s(x,y) - (a+b)| = |x + y - a - b| \leq |x - a| + |y - b| < 2\delta$$

We see for the sum proof we can choose $\delta = \epsilon/2$ and it will work out nicely.

Proof: Let $\epsilon > 0$ and let $(a,b) \in \mathbb{R}^2$. Choose $\delta = \epsilon/2$ and suppose $(x,y) \in \mathbb{R}^2$ such that $||(x,y) - (a,b)|| < \delta$. Observe that

$$||(x,y) - (a,b)|| < \delta \ \Rightarrow \ ||(x-a, y-b)||^2 < \delta^2 \ \Rightarrow \ |x-a|^2 + |y-b|^2 < \delta^2.$$

It follows $|x-a| < \delta$ and $|y-b| < \delta$. Thus

$$|s(x,y) - (a+b)| = |x + y - a - b| \leq |x - a| + |y - b| < \delta + \delta = 2\delta = \epsilon.$$

Therefore, $\lim_{(x,y)\to(a,b)} s(x,y) = a + b$. and it follows that the sum function if continuous at (a,b). But, (a,b) is an arbitrary point thus s is continuous on $\mathbb{R}^2$ hence the sum function is continuous. $\square$.

Preparing for the proof of continuity of the product function: I'll continue to use the same notation as above. We need to study $|p(x,y) - (ab)| = |xy - ab| < \epsilon$. Consider that

$$|xy - ab| = |xy - ya + ya - ab| = |y(x-a) + a(y-b)| \leq |y||x-a| + |a||y-b|$$

We know that $|x-a| < \delta$ and $|y-b| < \delta$. There is one less obvious factor to bound in the expression. What should we do about $|y|$?. I leave it to the reader to show that:

$$\boxed{|y-b| < \delta \qquad \Rightarrow \qquad |y| < |b| + \delta}$$

Now put it all together and hopefully we'll be able to "solve" for ϵ.

$$|xy - ab| = \leq |y||x-a| + |a||y-b| < (|b| + \delta)\delta + |a|\delta = \delta^2 + \delta(|a| + |b|) \text{ " } = \text{ " } \epsilon$$

I put solve in quotes because we have considerably more freedom in our quest for finding δ. We could just as well find δ which makes the " = " become an <. That said let's pursue equality,

$$\delta^2 + \delta(|a| + |b|) - \epsilon = 0 \qquad \delta = \frac{-|a| - |b| \pm \sqrt{(|a| + |b|)^2 + 4\epsilon}}{2}$$

Since $\epsilon, |a|, |b| > 0$ it follows that $\sqrt{(|a|+|b|)^2 + 4\epsilon} > \sqrt{(|a|+|b|)^2} = |a| + |b|$ hence the (+) solution to the quadratic equation yields a positive δ namely:

$$\boxed{\delta = \frac{-|a| - |b| + \sqrt{(|a| + |b|)^2 + 4\epsilon}}{2}}$$

Proof: Let $\epsilon > 0$ and let $(a,b) \in \mathbb{R}^2$. By the calculations that prepared for the proof we know that the following quantity is positive, hence choose

$$\delta = \frac{-|a| - |b| + \sqrt{(|a| + |b|)^2 + 4\epsilon}}{2} > 0.$$

Note that[3],

$$\begin{aligned} |xy - ab| = |xy - ya + ya - ab| &= |y(x-a) + a(y-b)| && \text{algebra} \\ &\leq |y||x-a| + |a||y-b| && \text{triangle inequality} \\ &< (|b| + \delta)\delta + |a|\delta && \text{by the boxed lemmas} \\ &= \delta^2 + \delta(|a| + |b|) && \text{algebra} \\ &= \epsilon \end{aligned}$$

where we know that last step follows due to the steps leading to the boxed equation in the proof preparation. Therefore, $\lim_{(x,y)\to(a,b)} p(x,y) = ab$. and it follows that the product function if continuous at (a,b). But, (a,b) is an arbitrary point thus p is continuous on $\mathbb{R}^2$ hence the product function is continuous. $\square$.

[3]my notation is that when we stack inequalities the inequality in a particular line refers only to the immediate vertical successor.

Proposition 3.2.11.

Let $\vec{F} : V \subseteq \mathbb{R}^p \to \mathbb{R}^m$ and $\vec{G} : U \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be mappings. Suppose that $\lim_{\vec{x}\to\vec{a}} \vec{G}(\vec{x}) = \vec{b}$ and suppose that $\vec{F}$ is continuous at $\vec{b}$ then

$$\lim_{\vec{x}\to\vec{a}} (\vec{F}\circ\vec{G})(\vec{x}) = \vec{F}\big(\lim_{\vec{x}\to\vec{a}} \vec{G}(\vec{x})\big).$$

One place to read the proof is in C.H. Edwards Advanced Calculus text, see pages 46-47. Notice that the proposition above immediately gives us the important result below:

Proposition 3.2.12.

Let $\vec{F}$ and $\vec{G}$ be mappings such that $\vec{F}\circ\vec{G}$ is well-defined. The composite function $\vec{F}\circ\vec{G}$ is continuous for points $\vec{a} \in dom(\vec{F}\circ\vec{G})$ such that the following two conditions hold:

1. $\vec{G}$ is continuous at $\vec{a}$
2. $\vec{F}$ is continuous at $\vec{G}(\vec{a})$.

I make use of the earlier proposition that a mapping is continuous iff its component functions are continuous throughout the examples that follow. For example, I know (Id, Id) is continuous since Id was previously proved continuous.

Example 3.2.13. *Note that if $f = p\circ(Id, Id)$ then $f(x) = \big(p\circ(Id, Id)\big)(x) = p\big((Id, Id)(x)\big) = p(x, x) = x^2$. Therefore, the quadratic function $f(x) = x^2$ is continuous on $\mathbb{R}$ as it is the composite of continuous functions.*

Example 3.2.14. *Note that if $f = p\circ(p\circ(Id, Id), Id)$ then $f(x) = p(x^2, x) = x^3$. Therefore, the cubic function $f(x) = x^3$ is continuous on $\mathbb{R}$ as it is the composite of continuous functions.*

Example 3.2.15. *The power function is inductively defined by $x^1 = x$ and $x^n = xx^{n-1}$ for all $n \in \mathbb{N}$. We can prove $f(x) = x^n$ is continous by induction on n. We proved the $n = 1$ case previously. Assume inductively that $f(x) = x^{n-1}$ is continuous. Notice that*

$$x^n = xx^{n-1} = xf(x) = p(x, f(x)) = (p\circ(Id, f))(x).$$

Therefore, using the induction hypothesis, we see that $g(x) = x^n$ is the composite of continuous functions thus it is continuous. We conclude that $f(x) = x^n$ is continuous for all $n \in \mathbb{N}$.

We can play similar games with the sum function to prove that sums of power functions are continuous. In your homework you will prove constant functions are continuous. Putting all of these things together gives us the well-known result that polynomials are continuous on $\mathbb{R}$.

Proposition 3.2.16.

Let a be a limit point of mappings $f, g : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}$ and suppose $c \in \mathbb{R}$. If $\lim_{x\to a} f(x) = b_1 \in \mathbb{R}$ and $\lim_{x\to a} g(x) = b_2 \in \mathbb{R}$ then

1. $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
2. $\lim_{x\to a}(f(x)g(x)) = \big(\lim_{x\to a} f(x)\big)\big(\lim_{x\to a} g(x)\big)$.
3. $\lim_{x\to a}(cf(x)) = c \lim_{x\to a} f(x)$.

Moreover, if f, g are continuous then $f + g$, fg and cf are continuous.

Proof: C.H. Edwards Advanced Calculus text proves (1.) carefully on pg. 48. I'll do (2.) here: we are given that If $\lim_{x\to a} f(x) = b_1 \in \mathbb{R}$ and $\lim_{x\to a} g(x) = b_2 \in \mathbb{R}$ thus by Proposition 3.2.5 we find $\lim_{x\to a}(f, g)(x) = (b_1, b_2)$. Consider then,

$$\begin{aligned}
\lim_{x\to a}(f(x)g(x)) &= \lim_{x\to a}\big(p(f,g)\big) && \text{defn. of product function} \\
&= p\big(\lim_{x\to a}(f,g)\big) && \text{since } p \text{ is continuous} \\
&= p(b_1, b_2) && \text{by Proposition 3.2.5.} \\
&= b_1 b_2 && \text{definition of product function} \\
&= \big(\lim_{x\to a} f(x)\big)\big(\lim_{x\to a} g(x)\big).
\end{aligned}$$

In your homework you proved that $\lim_{x\to a} c = c$ thus item (3.) follows from (2.). $\square$.

The proposition that follows does follow immediately from the proposition above, however I give a proof that again illustrates the idea we used in the examples. Reinterpreting a given function as a composite of more basic functions is a useful theoretical and calculational technique.

Proposition 3.2.17.

Assume $f, g : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}$ are continuous functions at $a \in U$ and suppose $c \in \mathbb{R}$.

1. $f + g$ is continuous at a.
2. fg is continuous at a
3. cf is continuous at a.

Moreover, if f, g are continuous then $f + g$, fg and cf are continuous.

Proof: Observe that $(f + g)(x) = (s \circ (f, g))(x)$ and $(fg)(x) = (p \circ (f, g))(x)$. We're given that f, g are continuous at a and we know s, p are continuous on all of $\mathbb{R}^2$ thus the composite functions $s \circ (f, g)$ and $p \circ (f, g)$ are continuous at a and the proof of items (1.) and (2.) is complete. To prove (3.) I refer the reader to their homework where it was shown that $h(x) = c$ for all $x \in U$ is a continuous function. We then find (3.) follows from (2.) by setting $g = h$ (function multiplication commutes for real-valued functions). $\square$.

We can use induction arguments to extend these results to arbitrarily many products and sums of power functions.To prove continuity of algebraic functions we'd need to do some more work with quotient and root functions. I haven't delved into the definition of exponential or log functions not to mention sine or cosine. However, we have shown those basic functions of calculus are continuous on the interior of their respective domains in the single-variable calculus course. Basically if the formula for a function can be evaluated at the limit point then the function is continuous.

It's not hard to see that the comments above extend to functions of several variables and mappings. If the formula for a mapping is comprised of finite sums and products of power functions then we can prove such a mapping is continuous using the techniques developed in this section. If we have a mapping with a more complicated formula built from elementary functions then that mapping will be continuous provided its component functions have formulas which are sensibly calculated at the limit point. In other words, if you are willing to believe me that $\sin(x), \cos(x), e^x, \ln(x), \cosh(x), \sinh(x), \sqrt{x}, \frac{1}{x^n}, \dots$ are continuous on the interior of their domains then it's not hard to prove:

$$f(x,y,z) = \left(\sin(x) + e^x + \sqrt{\cosh(x^2) + \sqrt{y+e^x}},\ \cosh(xyz),\ xe^{\sqrt{x+\frac{1}{yz})}}\right)$$

is a continuous mapping at points where the radicands of the square root functions are nonnegative. It wouldn't be very fun to write explicitly but it is clear that this mapping is the Cartesian product of functions which are the sum, product and composite of continuous functions.

3.3 multivariate indeterminants

Explicit calculations which show multivariate limits do not exist are important to think about. They bring added understanding to the constructions we've thus far endured. Also, from the perspective of the student, they are important since they are common test[4] questions.

I can summarize the results of the last section with a simple slogan: *the limit of a multivariate function is given by evaluation at the limit point provided the evaluation does not violate the laws of arithmetic*. In other words, you can solve the limit by just plugging in the limit point if there is no division by zero, even root of a negative number and/or inputs outside the domain of the elementary functions. We offer some methods to evaluate indeterminant limits and we will see how to show the limit does not exist.

In single variable calculus we learned that the double-sided limit exists iff both the left and right limits exist and are equal. However, consider $f : dom(f) \subseteq \mathbb{R}^2 \to \mathbb{R}$. If we look at a limit point (a,b) then there are infinitely many paths in $dom(f)$ which approach the limit point. Suppose we

[4]I do talk about things which are not on the test because this course is more than the tests or the homework, those are just tools to get you to start thinking, those are not the end, there is no end, this is the essence of what university education should be, an invitation to think, not just a path to a degree.

have a path $t \mapsto \vec{r}(t)$ with $\vec{r}(0) = (a, b)$ and $\lim_{t\to 0} \vec{r}(t) = (a, b)$. Then, for a real number L,

$$\boxed{\lim_{(x,y)\to(a,b)} f(x, y) = L \quad \Rightarrow \quad \lim_{t\to 0} f(\vec{r}(t)) = L.}$$

A direct consequence of the implication boxed above is that if we obtain different limits for two different paths through (a, b) then the limit of $f(x, y)$ as $(x, y) \to (a, b)$ does not exist. The examples below show how to pragmatically and orderly use this boxed equation to show limits fail to exist. I found these examples in Thomas' Calculus, which, depending on which edition you look at, can be an excellent text.

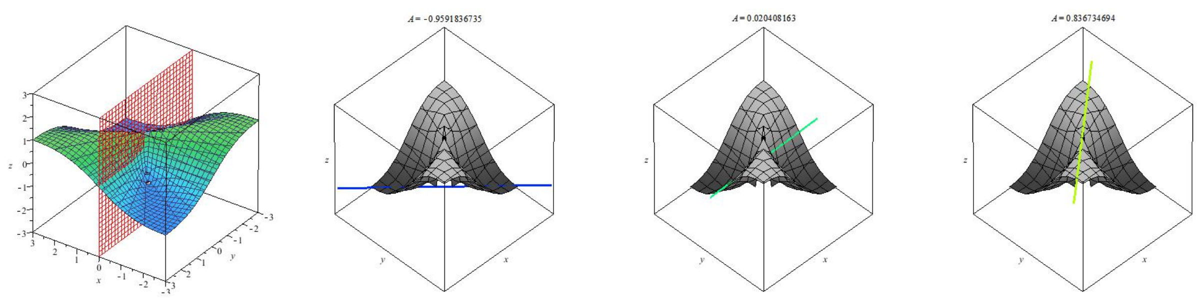
Example 3.3.1. *Suppose* $f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$. *Notice that we can calculate the limit for* $(a, b) \neq (0, 0)$ *with ease:*

$$\lim_{(x,y)\to(a,b)} = \frac{2ab}{a^2 + b^2}.$$

However, if we consider the limit at $(0, 0)$ *it is indeterminant since we have an expression of type* $0/0$. *Other calculation is required. Consider the path* $\vec{r}(t) = (t, mt)$ *then clearly this is continuous at* $t = 0$ *and* $\vec{r}(0) = (0, 0)$*; in-fact, this is just the parametric equation of a line* $y = mx$. *Consider,*

$$\lim_{t\to 0} f(\vec{r}(t)) = \lim_{t\to 0} \frac{2mt^2}{t^2 + m^2t^2} = \lim_{t\to 0} \frac{2m}{1 + m^2} = \frac{2m}{1 + m^2}.$$

The proposed limit $L = \frac{2m}{1+m^2}$ *depends nontrivially on* m *which means that paths with different* m *yield different limits. For example,* $m = 1$ *suggests* $L = 1$ *whereas* $m = -1$ *yields* $L = -1$. *It follows that the limit does not exist. Here's the graph of this function, maybe you can see the problem at the origin. The red plane is vertical through the origin. The three pictures on the right illustrate how differing linear paths yield differing limits*[5]



Curious, what if all linear-paths through the limit point yield the same limiting value? Is that a sufficient criteria to obtain a converse to the boxed implication? It seems plausible, if we look at all lines through the limit point then we've covered a neighborhood of the limit point with our analysis. Seems like we have a real chance. Well, until you study the next example:

[5] I'm using the function in Maple called "zhue" which colors an object corresponding to it's z-values.

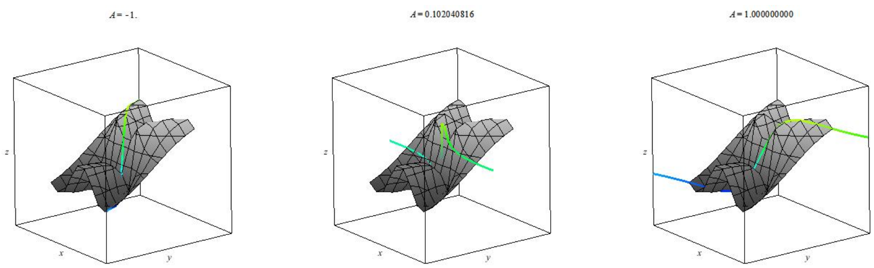
Example 3.3.2. *Suppose* $f(x,y) = \begin{cases} \frac{2x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$. *Notice that we can calculate the limit for* $(a,b) \neq (0,0)$ *with ease:*

$$\lim_{(x,y)\to(a,b)} = \frac{2a^2b}{a^4+b^2}.$$

However, if we consider the limit at $(0,0)$ *it is indeterminant since we have an expression of type* $0/0$*. Other calculation is required. Consider the path* $\vec{r}(t) = (t, mt)$ *then clearly this is continuous at* $t = 0$ *and* $\vec{r}(0) = (0,0)$*; in-fact, this is just the parametric equation of a line* $y = mx$*. Consider, for* $m \neq 0$*,*

$$\lim_{t\to 0} f(\vec{r}(t)) = \lim_{t\to 0} \frac{2mt^4}{t^4+m^2t^2} = \lim_{t\to 0} \frac{2mt^2}{t^2+m^2} = \frac{2m(0)}{0+m^2} = 0.$$

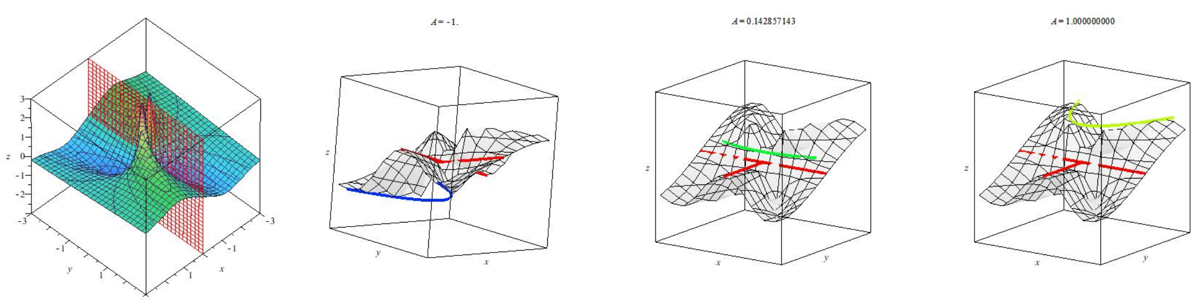
If $\vec{r}(t) = (t,0)$ *then for* $t \neq 0$ *we have* $f(\vec{r}(t)) = f(t,0) = 0$ *thus the limit of the function restricted to any linear path is just zero. The three pictures on the right illustrate how differing linear paths yield the same limits. The red lines are the* x, y *axes.*



What about parabolic paths? Those are easily constructed via $\vec{r}_2(t) = (t, kt^2)$ *again* $\vec{r}_2(0) = (0,0)$ *and* $\lim_{t\to 0}\vec{r}_2(t) = (0,0)$*. Calculate, for* $k \neq 0$*,*

$$\lim_{t\to 0} f(\vec{r}_2(t)) = \lim_{t\to 0} \frac{2kt^4}{t^4+k^2t^4} = \lim_{t\to 0} \frac{2k}{1+k^2} = \frac{2k}{1+k^2}.$$

Clearly if we choose differing values for k *we obtain different values for the limit hence the limit of* f *does not exist as* $(x,y) \to (0,0)$*. Here's the graph of this function, maybe you can see the problem at the origin. The red plane is vertical through the origin. The three pictures on the right illustrate how differing parabolic paths yield differing limits. The red lines are the* x, y *axes.*



I believe if we knew that the limit of $f \circ \vec{r}$ existed and were equal to L for all possible continuous paths then the multivariate limit of f would exist. However, I have not included proof in these notes at the present time. If you can prove or disprove this claim I'd be interested. It turns out that a multivariate function which is differentiable is consqeuently continuous. Thus, in the next chapter we find another tool for indirectly analyzing continuity. In any event, I hope this pair of examples gives you the idea. Moreover, while I have illustrated the concept for $f : \mathbb{R}^2 \to \mathbb{R}$ the techniques above equally well apply to indeterminant limits of $f : \mathbb{R}^3 \to \mathbb{R}$ or with a bit more imagination $f : \mathbb{R}^n \to \mathbb{R}$.

If you find the calculations of this last pair of examples a bit disheartening then you have my sympathy. Fortunately, there is another way to look at these examples and the technique will bring us to examples which are both indeterminant in their initial formulation and finite once the indeterminant form is resolved. The trick is coordinate change. Can we trade x, y for new coordinates which simplify the expression? Polar coordinates suffice for the examples above.

Example 3.3.3. *Suppose* $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$. *I argue it is intuitively clear the substitution below is valid:*

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \lim_{r\to 0} \frac{2r^2\cos\theta\sin\theta}{r^2} = \lim_{r\to 0}\bigg(2\cos\theta\sin\theta\bigg).$$

Therefore, the limit as $(x,y) \to (0,0)$ of f does not exist since as $r \to 0$ the function tends to $2\cos\theta\sin\theta$ which is not single-valued at the origin. What value of θ is assigned to the origin in polar coordinates?

You might complain that I am using polar coordinates precisely where they fail to be defined. However, I would argue it's reasonable since the limiting process considers points near the limit point but not the limit point itself. Polar coordinates are uniquely defined for points near the origin, just not the origin itself because the angle is infinitely-many valued at the origin. In practice we don't face trouble from this for a variety of reasons, but it is a fact that the origin is not labled in the same way as all the other points in the plane by polar coordinates.

Example 3.3.4. *Suppose* $f(x,y) = \begin{cases} \frac{2x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$. *Again, we use polar coordinate substitution,*

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \lim_{r\to 0} \frac{2r^3\cos^2\theta\sin\theta}{r^4\cos^4\theta + r^2\sin^2\theta} = \lim_{r\to 0} \frac{2r\cos^2\theta\sin\theta}{r^2\cos^4\theta+\sin^2\theta}.$$

If we choose $\theta = \pi/4$ then the limit above clearly tends to zero however if we consider the spiral path $\theta = r$ then we have $\frac{2r\cos^2 r\sin r}{r^2\cos^4 r+\sin^2 r} \to \frac{2rr}{r^2+r^2} = 1$. Therefore, the limit does not exist. This example is a bit subtle in any coordinate system, notice that looking at various choices for $\theta = const$ corresponds to sorting through lines of various slope. All of these linear paths, or rays in our current context, lead to the apparent triviality of the limit.

If there is no θ-dependence after changing to polar coordinates then the analysis simplifies.

Example 3.3.5. *Suppose* $f(x,y) = \begin{cases} \frac{x^2+y^2}{x^4+2x^2y^2+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$. *Again, we use polar coordinate substitution,*

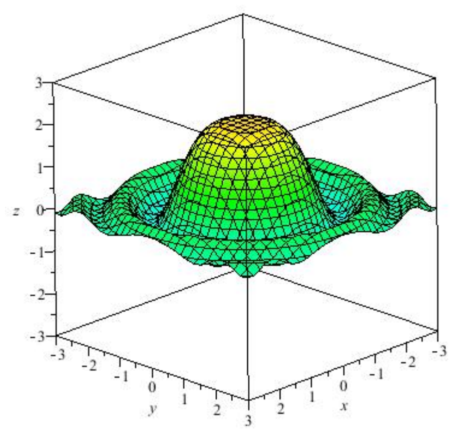
$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^4+2x^2y^2+y^4} = \lim_{r\to 0} \frac{r^2}{r^4} = \lim_{r\to 0} \frac{1}{r^2} = \infty.$$

this limit **diverged** *to* ∞*. We could see evidence of this in the graph, although infinities in graphs should be treated with great caution.*

Example 3.3.6. *Suppose* $f(x,y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$. *Again, we use polar coordinate substitution,*

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \underbrace{\lim_{r\to 0} \frac{\sin(r^2)}{r^2} = \lim_{r\to 0} \frac{2r\cos(r^2)}{2r}}_{L'Hospital's\ Rule\ on\ 0/0} = \lim_{r\to 0} \cos(r^2) = 1.$$

The graph agrees with our result. (rescaled for ease of viewing)



Naturally, we can also apply this technique to functions which admit a simplification in terms of spherical coordinates.

Example 3.3.7. *Suppose* $f(x,y,z) = \begin{cases} \sqrt{x^2+y^2+z^2}\,\ln(x^2+y^2+z^2) & (x,y) \neq (0,0,0) \\ 0 & (x,y,z) = (0,0,0) \end{cases}$. *Use spherical coordinate substitution,*

$$\begin{aligned}
\lim_{(x,y,z)\to(0,0,0)} \left[\sqrt{x^2+y^2+z^2}\,\ln(x^2+y^2+z^2) \right] &= \lim_{\rho\to 0} \left[\rho \ln(\rho^2) \right] \\
&= \lim_{\rho\to 0} \left[\frac{2\ln(\rho)}{1/\rho} \right] \qquad \text{L'Hospital's Rule on } \frac{\infty}{\infty}. \\
&= \lim_{\rho\to 0} \left[\frac{2/\rho}{-1/\rho^2} \right] \\
&= \lim_{\rho\to 0} \left[-2\rho \right] \\
&= 0.
\end{aligned}$$

The idea of substitution need not be limited to standard coordinate systems. Perhaps you'll find a challenge problem in the homework.

3.3.1 additional examples

More of the same here, but perhaps these help.

Example 3.3.8. .

$\boxed{E46}$ Let $f(x,y) = x^2 + \sqrt{y} + \tan^{-1}(x) + 3$, find limit at $(0,1)$

$$\lim_{(x,y)\to(0,1)} \left(x^2 + \sqrt{y} + \tan^{-1}(x) + 3 \right) = \underbrace{0 + \sqrt{1} + \tan^{-1}(0) + 3}_{\substack{\text{all the functions} \\ \text{involved are well behaved} \\ \text{at } x=0,\ y=1.}} = \boxed{4}$$

Example 3.3.9. .

Find the limit, if it exists.

$$\lim_{(x,y)\to(5,-2)} \left(x^5 + 4x^3y - 5xy^2 \right) = 5^5 - 45^3(-2) - 5(5)(4) = \boxed{2025}$$

this is the simple case where the function is continuous at the limit point. This means $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. Usually we cannot evaluate $f(a,b)$ and therein lies the difficulty, and the utility of the limit. Anyway, functions are continuous where they make sense.

Example 3.3.10. .

Find the limit, if it exists

$$\lim_{(x,y)\to(0,0)}\left(\frac{xy\cos(y)}{3x^2+y^2}\right) \quad \text{well notice} \quad \left.\frac{xy\cos(y)}{3x^2+y^2}\right|_{(0,0)} = \frac{0}{0}$$

which is indeterminant, we could get no value, $\pm\infty$ or even a finite #. If the limit exists then for all possible continuous paths to zero we should have the same limit. Approach along "x=axis" $(x,0)\to(0,0)$,

$$\lim_{(x,0)\to(0,0)}\left(\frac{xy\cos(y)}{3x^2+y^2}\right) = \lim_{x\to 0}\left(\frac{0}{3x^2+0}\right) = 0.$$

Now approach $(0,0)$ along $y=x$,

$$\lim_{(x,x)\to(0,0)}\left(\frac{x^2\cos(x)}{4x^2}\right) = \lim_{x\to 0}\left(\frac{\cos(x)}{4}\right) = \frac{\cos(0)}{4} = \frac{1}{4}$$

Thus we find different limits along different paths approaching $(0,0)$ therefore the $\boxed{\text{limit does not exist}}$