

Multi Mean Value Theorem $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$:

No, well, $f: U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ then

$$f(b) - f(a) = (\nabla f(c)) \cdot (b-a) = df_c(b-a)$$

Can't do this for $f: U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$, $m > 1$

However, $|f(b) - f(a)| \leq \|b-a\| \max_{x \in L} \|\nabla f(x)\|$ does generalize.

Definition: $|x|_0 = \max\{|x_1|, |x_2|, \dots, |x_n|\}$

$$C_r^n = \{x \in \mathbb{R}^n \mid |x|_0 \leq r\}$$

$$\partial C_r^n = \{x \in \mathbb{R}^n \mid |x|_0 = r\}$$

Definition: $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$, norm of L , call it $\|L\|$

$$\|L\| = \max_{x \in \partial C_1} |L(x)|_0$$

Why does $\|L\|$ exist?

$x \rightarrow L(x)$ continuous b/c L is linear

$x \rightarrow |x|_0$ also continuous

But ∂C_1 is compact \Rightarrow max is obtained.

$$\left| \frac{x}{|x|_0} \right|_0 = 1, \quad \frac{x}{|x|_0} \in \partial C_1$$

$$|L\left(\frac{x}{|x|_0}\right)|_0 \leq \|L\| \Rightarrow \left| \frac{1}{|x|_0} L(x) \right|_0 \leq \|L\| \Rightarrow \frac{1}{|x|_0} |L(x)|_0 \leq \|L\|$$

$$\Rightarrow |L(x)|_0 \leq |x|_0 \|L\|, \quad x \in \mathbb{R}^n$$

Prop 2.11 $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$, then $\|L\|$ is least # M s.t.
 $|L(x)|_0 \leq M|x|_0 \quad \forall x \in \mathbb{R}^n$

Proof: Suppose $\exists M$ s.t. $M < \|L\|$ and $|L(x)|_0 \leq M|x|_0$
 $\Rightarrow \forall x \in \delta C_1, |L(x)|_0 \leq M$

$\Rightarrow M$ is an upper bound of $\{|L(x)|_0 \mid x \in \delta C_1\}$

but $\|L\|$ is sup $\{|L(x)|_0 \mid x \in \delta C_1\}$

$\Rightarrow M \geq \|L\|$

$\Rightarrow (!)$

\Rightarrow Done!

Lemma: If $L = (L_1, L_2, L_3, \dots, L_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear, then

$$\|L_i\| \leq \|L\| \quad \forall i = 1, 2, \dots, m$$

Proof: $\|L_i\| = |L_i(x_0)| \leq \max \{L_1(x_0), L_2(x_0), \dots, L_m(x_0)\} = |L(x_0)|_0$
 $\leq \max |L(x)|_0 = \|L\|$.

Definition: $A \in \mathbb{R}^{m \times n}$

$$\|A\| = \max_{1 \leq i \leq m} \left(\sum_{j=1}^n |A_{ij}| \right) = \max \{ |\text{row}_1(A)|, |\text{row}_2(A)|, \dots, |\text{row}_m(A)| \}$$
$$\approx \|x\|_1 = \sum_{i=1}^n |x_i| \text{ (one-norm)}$$

Ex: $L: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$L(x, y, z) = (x - 3z, 2x - y - 2z, x + y) = \begin{pmatrix} 1 & 0 & -3 \\ 2 & -1 & -2 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\|A\| = \max \{ 4, 5, 2 \} = 5$$

Theorem 2.3 $A \in \mathbb{R}^{n \times n}$, $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ w/ $L(x) = Ax \forall x \in \mathbb{R}^n$

$$\Rightarrow \|L\| = \|A\|$$

Proof: $|L(x)|_0 = \max_{1 \leq i \leq m} \left\{ \left| \sum_{j=1}^n A_{ij} x_j \right| \right\}$

$$= \left| \sum_{j=1}^n A_{kj} x_j \right|, \text{ some } k$$

$$\leq \sum_{j=1}^n |A_{kj}| |x_j| \leq \sum_{j=1}^n |A_{kj}| |x|_0$$

$$= |x|_0 \sum_{j=1}^n |A_{kj}| \leq |x|_0 \max_{1 \leq i \leq m} \left\{ \sum_{j=1}^n |A_{ij}| \right\} = |x|_0 \|A\|$$

$$\Rightarrow \|L\| \leq \|A\| \text{ by Prop. 2.1}$$

$\|L\| \geq \|A\|$ pick k^{th} row w/ $\|A\| = \sum_{j=1}^n |A_{kj}|$

For $j=1, \dots, n$ defined $\varepsilon_j = \pm 1$ where $A_{kj} = \varepsilon_j |A_{kj}|$

If $x = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ then $|x|_0 = 1$ and

$$\begin{aligned} \|L\| \geq \|L(x)\|_0 &= \max \left| \sum_{j=1}^n A_{kj} \varepsilon_j \right| \geq \left| \sum_{j=1}^n A_{kj} \varepsilon_j \right| = \left| \sum |A_{kj}| \right| \\ &= \sum |A_{kj}| \\ &= \|A\| \end{aligned}$$

$$\Rightarrow \|L\| \geq \|A\|$$

$$\therefore \|L\| = \|A\|$$

$$\Phi: L(\mathbb{R}^n, \mathbb{R}^m) \longrightarrow \mathbb{R}^{m \times n}$$

$$\|\Phi(L)\| = \|L\|$$

$$L(\mathbb{R}^n, \mathbb{R}^m) \stackrel{\mathcal{L}_{mn}}{\cong} \mathbb{R}^{m \times n}$$

If $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at x then

$$df_x \in \mathcal{L}_{mn} \text{ and } f'(x) \in \mathbb{R}^{m \times n}$$

hence $f' : (x \mapsto f'(x))$

$$f': \mathbb{R}^n \longrightarrow \mathbb{R}^{m \times n}$$

$$df: \mathbb{R}^n \longrightarrow \mathcal{L}_{mn}$$

Definition: $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is C^1 at $a \in \mathbb{R}^n$

$\Leftrightarrow \partial_i f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ are $C^0(a)$ for $i=1, 2, \dots, n$

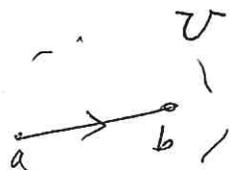
$\Leftrightarrow f$ continuously differentiable at a

Proposition. Let diff. map $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is in $C^1(a)$ iff $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous at a .

Proof:

Theorem: $f: U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a C^1 mapping, and

then $\|f(b) - f(a)\|_0 \leq \|b - a\|_0 \max_{x \in L} \|f'(x)\|$



Proof: $h = b - a$. Define $\gamma: [0, 1] \rightarrow \mathbb{R}^m$

$$\gamma(t) = f(a + th) \quad [\gamma(0) = f(a), \gamma(1) = f(b)]$$

$$\gamma_i(t) = f^i(a + th)$$

$$\gamma_i'(t) = d f_{a+th}^i(h)$$

$$\|f(b) - f(a)\|_0 = \|f^k(b) - f^k(a)\| = \|\gamma_k(1) - \gamma_k(0)\|$$

\exists such a k

$$= \left| \int_0^1 \left(\frac{d\gamma_k}{dt} \right) dt \right| : \text{(FTC)}$$

$$\leq \|h\|_0 \cdot \|df_{a+th}^k\| \text{ max for } t = \tau$$

$$\leq \|h\|_0 \cdot \max_{t \in (0,1)} \|df_{a+th}^k\|$$

$$= \|h\|_0 \max_{x \in L} \|f'(x)\|$$

$$\leq \int_0^1 \|\gamma_k'(t)\| dt$$

$$= \int_0^1 \|df_{a+th}^k(h)\| dt$$

$$\leq \max_{t \in [0,1]} \|df_{a+th}^k(h)\|$$

$$\leq \|h\|_0 \cdot \max_{t \in (0,1)} \|df_{a+th}^k\| \text{ prop. 2.1}$$

Cor 2.6] $f: U \rightarrow \mathbb{R}^m$ is C^1 & $U \subseteq \mathbb{R}^n$

Containing line L w/ end pts $a, a+h$

If $\lambda: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear then

$$|f(a+h) - f(a) - \lambda(h)|_0 \leq |h|_0 \max_{x \in L} \|df_x - \lambda\|$$

Cor 2.7 $U \subseteq \mathbb{R}^n$ and $C_r \subset U$ and $f: U \rightarrow \mathbb{R}^m$ is a C^1 mapping

w/ $f(0) = 0$ and $df_0 = I$. If

$$\|df_x - I\| < \varepsilon \text{ for all } x \in C_r$$

then $f(C_r) \subset C_{(1+\varepsilon)r}$

Proof: set $a = 0$, $\lambda = df_0 = I$, $h = x \in C_r$

$$|f(x) - x| \leq |x|_0 \max_{x \in L} \|df_x - I\| < |x|_0 \varepsilon.$$

$$||f(x)|_0 - |x|_0| < |f(x) - x| < |x|_0 \varepsilon < \varepsilon r$$

$$\Rightarrow |f(x)|_0 < \varepsilon(r+1)$$

$$\Rightarrow C_{(1+\varepsilon)r}$$

Defn $\varphi: C \rightarrow C$, $C \subseteq \mathbb{R}^n$ is contraction map if $\|\varphi(x) - \varphi(y)\|_0 \leq k \|x - y\|_0 \quad \forall x, y \in C$

Theorem: If $\varphi: C \rightarrow C$ is contraction mapping w/ $k < 1$ and C is closed and bounded subset of \mathbb{R}^n

Then φ has unique fixed pt. x_* which for $x_0 \in C$ is given as the limit of $\{x_n\}_{n=0}^{\infty}$ where $x_{n+1} = \varphi(x_n)$ also, $\|x_m - x_*\|_0 \leq k^m \frac{\|x_0 - x_1\|_0}{1-k}$

$f: \mathbb{R}^n \rightarrow \mathbb{R}^n$. $f \in C^1(a)$

f is locally invertible at a if df_a is invertible.

$$f^1(\vec{x}) = y_1$$

$$f^2(\vec{x}) = y_2, \quad f(x) = y, \text{ solve for } x$$

$$f^n(\vec{x}) = y_n$$

$$df_a^{-1}(x) = y_1$$

$$df_a^{-2}(x) = y_2$$

$$df_a^{-n}(x) = y_n$$

unique
solⁿ for

why df^{-1} exists is necessary given existence of g
s.t $f \circ g = id_v$, $g \circ f = id_v$

Ex: Counter example to df^{-1} exists on $V \Rightarrow f|_U$ is 1-1

$$f(x, y) = (x^2 - y^2, 2xy)$$

$$\cos^2 \theta - \sin^2 \theta = \cos 2\theta$$

$$2 \sin \theta \cos \theta = \sin 2\theta$$

$$f(r \cos \theta, r \sin \theta) = (r^2 \cos 2\theta, r^2 \sin 2\theta)$$

$$f^{-1}\{(r^2 \cos 2\theta, r^2 \sin 2\theta)\} = \{(r \cos \theta, r \sin \theta),$$

$$(r \cos(\theta + \pi), r \sin(\theta + \pi))\}$$
$$f'(x, y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix} = 4(x^2 + y^2) \neq 0 \text{ for } \mathbb{R}^2 - \{(0, 0)\}$$

Lemma 3.2: Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ w/ $f(0) = 0$ and $df_0 = I$

$$\beta \quad \|df_x - I\| \leq \varepsilon < 1 \quad \forall x \in C_r \text{ within } r\text{-units of } 0$$
$$|x_0|_0 \leq r$$

Then $C_{(1-\varepsilon)r} \subset f(C_r) \subset C_{(1+\varepsilon)r}$. Moreover, if $V = \text{int } C_{(1-\varepsilon)r}$

$$\text{and } U = \text{int}(C_r \cap f^{-1}(V))$$

then $f: U \rightarrow V$ is 1-1 and onto.

and the inverse $g: V \rightarrow U$ is differentiable at 0 .

Finally, the local g is limit of $\{g_m\}_0^\infty$ where

$$g_0(y) = 0 \quad \& \quad g_{m+1}(y) = g_m(y) - f(g_m(y)) + y$$

Proof: Cor. 2.7 $f(C_r) \subset C_{(1+\epsilon)r}$ and it follows

from proof of Cor. 2.8 that f is 1-1 on C_r .

Cor. 2.6 w/ $\lambda = df_0 = I$ to see that

$$|f(x) - f(y) - (x-y)|_0 \leq \epsilon |x-y|_0 \quad (*)$$

for $x, y \in C_r$. It follows, $f(C_r) \subset C_{r(1+\epsilon)}$
~~image under~~

$$(1-\epsilon) |x-y|_0 \leq |f(x) - f(y)|_0 \leq (1+\epsilon) |x-y|_0$$

$\Rightarrow f$ is 1-1 on C_r

Seek to show $C_{(1-\epsilon)r} \subset f(C_r)$. Given $y \in C_{(1-\epsilon)r}$

$$\text{Define } \varphi_y(x) = x - f(x) + y$$

$$\varphi_y: C_r \rightarrow C_r$$

Apply Corollary 2.6, show φ is onto C_r , $x \in C_r$, $|x|_0 \leq r$.

$$|\varphi(x)|_0 \leq |x - f(x) + y|_0 \leq |f(x) - x|_0 + |y|_0$$

$$= |f(x) - f(0) - df_0(x-0)|_0 + |y|_0$$

$$\leq |x|_0 \max_{\substack{x \in C_r \\ x \in C_r}} \|df_x - df_0\| + |y|_0$$

$$= \cancel{r\epsilon} + (1-\epsilon)r = r$$

$$\rho(x) - \varphi(z) \Big|_0 \leq |x - z - (f(x) - f(z))|_0$$

$$\leq \varepsilon |x - z|_0 \quad (\text{b/c } (*))$$

$\Rightarrow \varphi$ is contraction mapping

$\Rightarrow \exists!$ fixed point x^*

$y \in V = \text{int}(C_{(1-\varepsilon)r})$ then the fixed point for φ is inside C_r

\therefore if $U = f^{-1}(V) \cap \text{int}(C_r)$ then $U \notin V$ are open neighborhood of zero,

$f: U \rightarrow V$ is 1-1 and onto.

The fact fixed pt $x = g(y)$ is limit of sequence

$$x_0 = 0, \quad x_{m+1} = x_m - f(x_m) + y$$

follows from contr. mapping theorem.

we need to show ~~that~~ is that g is diff at zero.

$$g(0) = 0$$

$$\lim_{h \rightarrow 0} \frac{\|g(h) - g(0) - dg(h)\|_0}{\|h\|_0} = \lim_{h \rightarrow 0} \frac{\|g(h) - h\|_0}{\|h\|_0}$$

$$\left. \begin{array}{l} y=0 \\ x=g(h) \\ h=f(h) \end{array} \right| |g(h)-h|_0 \leq \varepsilon |g(h)|_0 = \varepsilon |x|_0$$

$$(1-\varepsilon) |x|_0 \leq |f(x)|_0$$

$$|x|_0 \leq \frac{1}{1-\varepsilon} |f(x)|_0$$

$$\varepsilon |x|_0 \leq \frac{\varepsilon}{1-\varepsilon} |f(x)|_0$$

$$\Rightarrow |g(h)-h|_0 \leq \varepsilon |x|_0 \leq \frac{\varepsilon}{1-\varepsilon} |f(x)|_0 = \frac{\varepsilon}{1-\varepsilon} |h|_0$$

$$\Rightarrow \frac{|g(h)-h|_0}{|h|_0} \leq \frac{\varepsilon}{1-\varepsilon}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{|g(h)-h|_0}{|h|_0} = 0$$

Theorem 3.3 $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and C^1 in nhood W containing a
w/ matrix $f'(a)$ non-singular. Then f is locally
invertible, \exists nhoods $U \subset W$ of a and V of $b = f(a)$

and a 1-1 mapping $g: V \rightarrow W$ s.t

$$g(f(x)) = x \quad \forall x \in U \text{ and}$$

$$f(g(y)) = y \quad \text{for } y \in V$$

In particular, g is limit of $\{g_m\}_0^\infty$

$$g_0(y) = a, \quad g_{k+1}(y) = g_k(y) - (f'(g_k))^{-1} f(g_k(y))$$

for $y \in V$

Proof: 185 - 187 of Edwards.

Implicit Mapping Theorem

$G: \mathbb{R}^{m+n} \rightarrow \mathbb{R}^n$ (n equations in $(m+n)$ -variables
 $\Rightarrow n$ -unknowns)

$$x \in \mathbb{R}^m, y \in \mathbb{R}^n$$

$$(x, y) \in \mathbb{R}^{m+n}$$

$$\begin{cases} G_1(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n) = 0 \\ \vdots \\ G_n(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n) = 0 \end{cases}$$

Solution for y_1, y_2, \dots, y_n in terms of x_1, x_2, \dots, x_m

Goal: Find $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ s.t. $y = f(x)$ solves $G(x, y) = 0$

Special Notation: $P = (a, b)$ ($a \in \mathbb{R}^m \notin b \in \mathbb{R}^n$) $e \in \mathbb{R}^{m+n}$

$$d_x G_p(r) = dG_p(r, 0) \text{ has matrix } \frac{\partial G}{\partial x} = \left[\frac{\partial G}{\partial x_1} \mid \frac{\partial G}{\partial x_2} \mid \dots \mid \frac{\partial G}{\partial x_m} \right]$$

$$d_y G_p(s) = dG_p(0, s) \text{ has matrix } \frac{\partial G}{\partial y} = \left[\frac{\partial G}{\partial y_1} \mid \frac{\partial G}{\partial y_2} \mid \dots \mid \frac{\partial G}{\partial y_n} \right]$$

$$G' = \left[\frac{\partial G}{\partial x} \mid \frac{\partial G}{\partial y} \right] \in \mathbb{R}^{n \times (m+n)} \quad e \in \mathbb{R}^{m+n}$$

$x \in \mathbb{R}^m, y \in \mathbb{R}^n$ both differentiable function of t

$$x = \alpha(\vec{t}), \quad y = \beta(\vec{t})$$

$$\varphi = (\alpha, \beta) : \mathbb{R} \rightarrow \mathbb{R}^{m+n}$$

$$\varphi(\vec{t}) = G(\alpha(\vec{t}), \beta(\vec{t}))$$

$$\varphi'(\vec{t}) = (d_x G(\alpha(t), \beta(t)))(\alpha'(t)) + (d_y G(\alpha(t), \beta(t)))(\beta'(t))$$

$G: \mathbb{R}^{m+n} \rightarrow \mathbb{R}^n$ is diff. in nhood of (a, b) where $G(a, b) = 0$.

Furthermore, $\exists \theta(x, y) = 0 \Rightarrow y = h(x)$ for some differentiable function h , near a

$$h(a) = b \quad \& \quad G(x, h(x)) = 0$$

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \frac{dh}{dx} = 0$$

$$\Rightarrow h'(x) = \left(-\frac{\partial G}{\partial y} \right)^{-1} \frac{\partial G}{\partial x}$$

Theorem: $G: \mathbb{R}^{m+n} \rightarrow \mathbb{R}^n$ a C^1 function near (a, b) w/
 $G(a, b) = 0$, $\frac{\partial G}{\partial y}(a, b)$ is non-singular

and \exists nhood U of a in \mathbb{R}^m and W of $(a, b) \in \mathbb{R}^{m+n}$

and a C^1 mapping $h: U \rightarrow \mathbb{R}^n$ s.t

$y = h(x)$ solves $G(x, y) = 0$ in W .

h is limit of $\{h_k\}_0^\infty$ defined by

$$h_0(x) = b$$

$$h_{k+1}(x) = h_k(x) - \left(\frac{\partial G}{\partial y}(a, b) \right)^{-1} G(x, h_k(x))_0$$

proof: Let $f: \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m+n}$ be defined by

$$f(x, y) = (x, G(x, y)) = \begin{pmatrix} x \\ G \end{pmatrix}$$

$$f'(x, y) = \left[\frac{\partial f}{\partial x} \mid \frac{\partial f}{\partial y} \right] = \left[\begin{array}{c|c} I & 0 \\ \hline \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{array} \right]$$

$$\det(f'(x, y)) = \det\left(\frac{\partial G}{\partial y}\right) \neq 0 \text{ at } (a, b) \text{ by assumption.}$$

f is C^1 because

$(x, y) \rightarrow x$ is C^1

$(x, y) \rightarrow G(x, y)$ is given to be C^1

Cartesian product of C^1 maps is C^1

Hence $\exists U, V$ s.t. $f: U \rightarrow V$ is bijection

$\& g: V \rightarrow U$ is its diffeomorphism inverse

$$g(f(x, y)) = (x, y)$$

$$(g_1(f(x, y)), g_2(f(x, y))) = (x, y)$$

$$\cancel{F(x,y) = (x,y) = (a,b)}$$

Manifolds embedded in \mathbb{R}^n :

a. Implicitly (level sets)

or Explicitly (parametrization)

Ex: $L \subset \mathbb{R}^3$

1-parameter \Rightarrow 1dim'l manif

① $\vec{r} : \mathbb{R} \rightarrow L, \vec{r}(t) = (1, 2, 3) + t(4, 5, 6)$

②
$$\left. \begin{aligned} x &= 1 + 4t \\ y &= 2 + 5t \\ z &= 3 + 6t \end{aligned} \right\} \Rightarrow \frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6} = t$$

$$G(x, y, z) = \left(\frac{1}{4}(x-1) - \frac{1}{5}(y-2) + \frac{1}{6}(z-3) \right)$$

