Building Superalgebras from Supernumbers

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Motivations and Overview

- The path integral quantization requires the use of Grassmann variables. Commuting fields model bosons whereas anticommuting fields model fermions.

- Supersymmetry (SUSY) posits an equivalence between bosons and fermions, superspace provides a natural realization of this symmetry.

- There are physical reasons for using infinitely generated supernumbers. The approach we follow was pioneered by Alice Rogers.

- Our goal is to survey the program of geometric supermathematics. Along the way we will find how to construct certain infinite dimensional Lie algebras.

- Grassmann generators anticommute,

$$\zeta^i \zeta^j = -\zeta^j \zeta^i \qquad \zeta\zeta = 0$$

- Supernumbers are built with Grassmanns

$$Z = Z_o + Z_i \zeta^i + Z_{ij} \zeta^i \zeta^j + \cdots$$

- Λ is the set of supernumbers with finite norm

$$|Z| = |Z_o| + \sum |Z_i| + \sum |Z_{ij}| + \cdots$$

- Supernumbers break into even/odd parts; $\Lambda = {}^{0}\Lambda \oplus {}^{1}\Lambda$

$$z = \underbrace{\sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_{2p}} z_I \zeta^I}_{even} + \underbrace{\sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_{2p+1}} z_I \zeta^I}_{odd}$$

- Λ - valued variables are Grassmann variables.

- Flat superspace of dimension (p|q) provides p-commuting Grassmann variables and q-anticommuting variables

$$\begin{aligned} K^{p|q} &= ({}^{0}\Lambda)^p \times ({}^{1}\Lambda)^q \\ &= \left\{ (x^1, \ldots, x^p, \theta^1, \ldots, \theta^q) \mid x^i \in {}^{0}\Lambda,\ \theta^\alpha \in {}^{1}\Lambda \right\} \end{aligned}$$

Supersmooth functions

- Superspace is a Banach space thus one can consider Frechet derivatives and smooth functions.

- A function which is smooth and satisfies an additional linearity condition which respects the (p|q) structure is said to be supersmooth. A function is G^1 if

$$f(a+h,b+k) \approx f(a,b) + \sum_{m=0}^{p} h^m \left(\frac{\partial f}{\partial x^m}\right)(a,b) + \sum_{\alpha=0}^{q} k^\alpha \left(\frac{\partial f}{\partial \theta^\alpha}\right)(a,b)$$

An example of super calculus

- The function below is said to be “even” since its range lies in the even supernumbers, it has “parity” zero.

$$f(x, \theta, \beta) = x\theta\beta + 1$$

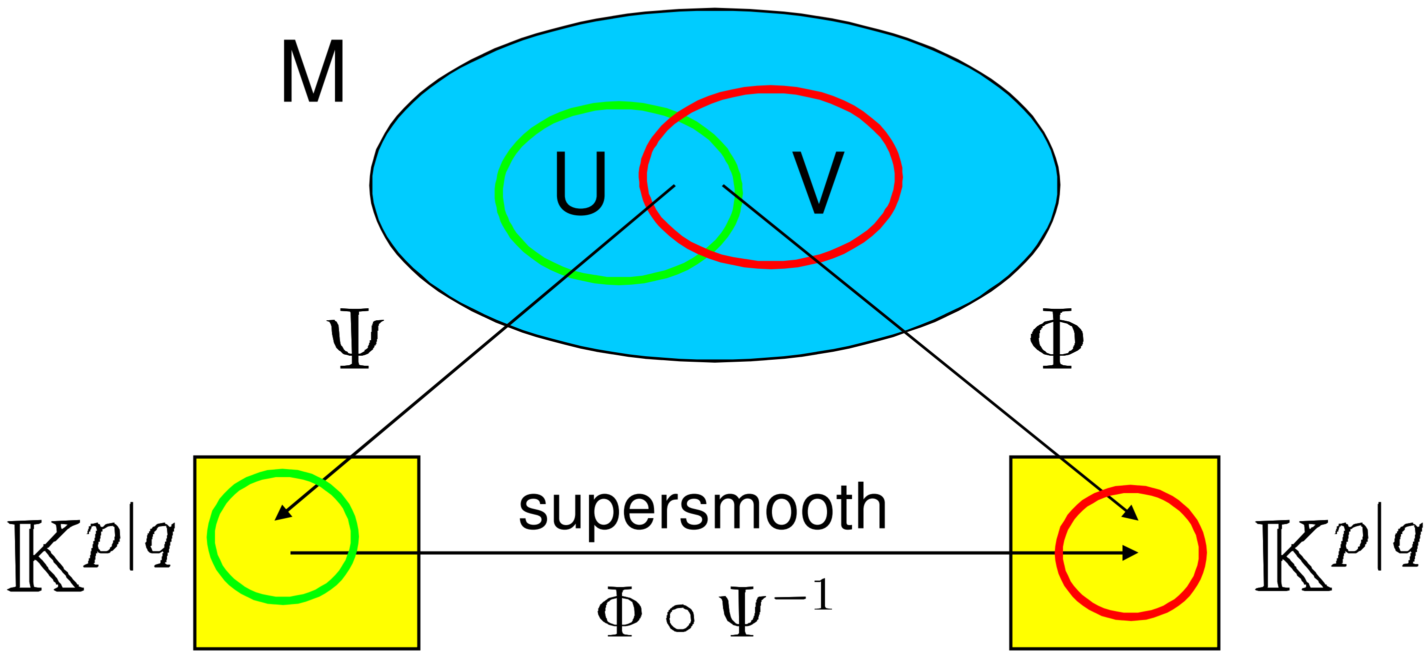
$$\frac{\partial f}{\partial x} = \theta\beta \qquad \frac{\partial f}{\partial \theta} = x\beta \qquad \frac{\partial f}{\partial \beta} = -x\theta$$

- Odd derivatives anticommute with odd functions. Moreover odd derivatives change the parity of the function while even derivatives do not.

$$\theta\beta = even \qquad x\beta = odd$$

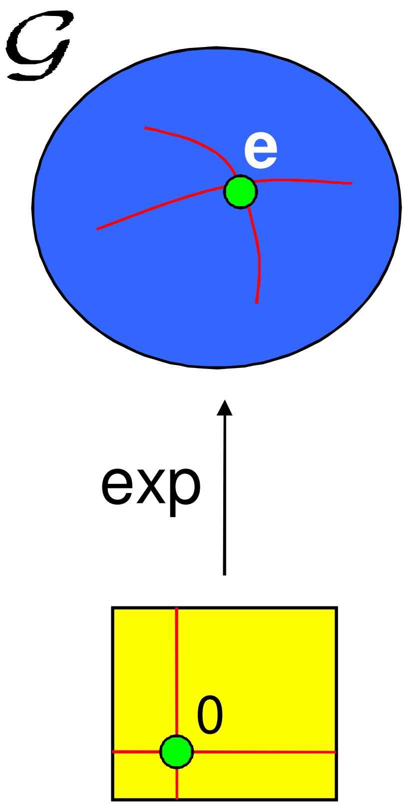
G^{∞} Supermanifold

- A (p|q) dimensional supermanifold M is locally $\mathbb{K}^{p|q}$, it has an atlas of G^{∞} charts



G^∞ Super Lie Group

- A supermanifold $\mathcal{G}$ which is also group with G^∞ operations is a super Lie group.



$$\mathfrak{g} = T_e^0\mathcal{G}$$

- Tangent Module at e is $T_e\mathcal{G}$

$T_e^0\mathcal{G}$	$T_e^1\mathcal{G}$

- even: $\partial/\partial x^m \in T_e^0\mathcal{G}$
- odd: $\partial/\partial\theta^\alpha \in T_e^1\mathcal{G}$
- No coordinate basis for $T_e^0\mathcal{G}$

Why exp only works for even objects

Suppose a,b $\in {}^1\Lambda$, notice $exp(a) = 1 + a$ and $exp(b) = 1 + b$. Likewise $exp(a + b) = 1 + a + b$. Now observe that,

$$exp(a)exp(b) = (1 + a)(1 + b) = 1 + a + b + ab \neq 1 + a + b = exp(a + b)$$

thus even in the one-dimensional case the exp is difficult to interpret.

- We must input even objects into exp. The even part of the tangent module is identified with the underlying Banach Lie algebra. The exponential is defined in terms of a flow on a Banach Lie Group.

$\mathbb{Z}_2$ - Graded Lie Algebras

A graded Lie algebra is a graded vector space U=U$_0 \oplus U_1$ over $\mathbb{K}$ with a bilinear bracket $[\ ,\ ] : U \times U \to U$ which is graded $[U_r, U_s] \subset U_{r+s}$ for $r, s = 0, 1$, and for all $a, b, c \in U_0 \cup U_1$ with parities $\epsilon_a, \epsilon_b, \epsilon_c$ satisfies the graded Jacobi indentity

$$(-1)^{\epsilon_a \epsilon_c}[a,[b,c]] + (-1)^{\epsilon_b \epsilon_a}[b,[c,a]] + (-1)^{\epsilon_c \epsilon_b}[c,[a,b]] = 0$$

and the graded skewsymmetry condition,

$$[a,b] = -(-1)^{\epsilon_a \epsilon_b}[b,a].$$

Graded Lie Algebra of Matrices

- Matrices in gl(m|n,K) form a graded vector space,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = M_0 + M_1 \ , \ M_0 = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \ M_1 = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

- Matrix multiplication is a graded multiplication,

$$gl^r(m|n,\mathbb{K})gl^s(m|n,\mathbb{K}) = gl^{r+s}(m|n,\mathbb{K})$$

- In fact these form a graded-Lie algebra with respect to the graded bracket defined below, $A_r \in gl^r(m,n,\mathbb{K})$

$$[A_r, B_s] = A_r B_s - (-1)^{\epsilon_r \epsilon_s} B_s A_r$$

Super Lie Algebra

A graded left Λ-module is a graded vector space $U = U_0 \oplus U_1$ which is a left module that respects the parity structures of U and Λ; that is ${}^0\Lambda U_r \subset U_r$ and ${}^1\Lambda U_r \subset U_{r+1}$ for $r \in \{0,1\} = \mathbb{Z}_2$.

A graded Lie left Λ-module is a graded Lie algebra W over $\mathbb{K}$ which is a left Λ-module such that for all $\alpha \in \Lambda$ and $X, Y \in W$

$$[\alpha X, Y] = \alpha[X, Y].$$

W is also called a Super Lie algebra.

- Z_2 - graded Lie algebras are sometimes called Super Lie algebras. However, we reserve the term for algebras built over the supernumbers. Much more is known in the graded case.

Super Lie Algebra of Matrices

- Each supernumber splits $z = z_c + z_a$ thus

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = M_0 + M_1 \ , \ M_0 = \begin{pmatrix} A_c & B_a \\ C_a & D_c \end{pmatrix} \ \ M_1 = \begin{pmatrix} A_a & B_c \\ C_c & D_a \end{pmatrix}$$

- Super matrix multiplication is a graded multiplication,

$$gl^r(m|n,\Lambda)gl^s(m|n,\Lambda) = gl^{r+s}(m|n,\Lambda)$$

- In fact these form a super Lie algebra with respect to the graded bracket defined below, $A_r \in gl^r(m|n,\Lambda)$

$$[A_r, B_s] = A_r B_s - (-1)^{\epsilon_r \epsilon_s} B_s A_r$$

Creating Super Lie Algebras from Graded Algebras

If $\mathfrak{g}_{Lie}$ is a graded Lie algebra over $\mathbb{C}$, then its Grassmann shell is the super Lie algebra $\widehat{g_{Lie}} = \Lambda \otimes g_{Lie}$ defined by

$$[\lambda X, \mu Y] = \lambda\mu(-1)^{\epsilon(\mu)\epsilon(X)}[X, Y]_{g_{Lie}}$$

for $\lambda, \mu \in \Lambda$ and $X, Y \in g_{Lie}$. More generally, one says that a super Lie algebra $\mathfrak{g}$ is a conventional Berezin superalgebra of dimension (p, q) if and only if it possesses a pure basis for which the structure constants have no soul.

- The Grassmann shell of $gl(m|n, \mathbb{C})$ is $gl(m|n, \Lambda)$.

An application of the graded Ado's Theorem

A generalization of Ado's Theorem due to Kac states that for any graded Lie algebra g_{Lie} there exists an even injective homomorphism of graded Lie algebras

$$\phi : g_{Lie} \hookrightarrow gl(r|s, \mathbb{C})$$

Notice that the injective homomorphism naturally extends to the Grassmann shell, thus we injectively embed the Grassmann shell of the graded Lie algebra into matrices having Grassmann supernumbers as entries:

$$\phi : \widehat{g_{Lie}} \hookrightarrow gl(r|s, \Lambda).$$

Supergroups corresponding to graded algebras

Let $\mathcal{G}$ denote a type $(p|q)$ dimensional super Lie group and $\mathfrak{g} = \mathcal{L}(\mathcal{G})$ its super Lie algebra of left invariant vector fields. Let $h \subseteq \mathfrak{g}$ be a (r, s) dimensional sub-super Lie algebra of $\mathfrak{g}$ which is closed and split in $\mathcal{L}(\mathcal{G})$. Then there is a type $(r|s)$ super Lie group $\mathcal{H}$ which is a subgroup of $\mathcal{G}$ such that $\mathcal{L}(\mathcal{H}) = h$ and the inclusion $i : \mathcal{H} \to \mathcal{G}$ is a G^∞ injective immersion. (Theorem 5.9 in our paper)

It can be shown that $Gl(m|n, \Lambda)$ is a super Lie group with $\mathcal{L}(Gl(m|n, \Lambda)) = gl(m|n, \Lambda)$. Moreover, $\widehat{g_{Lie}}$ is a closed and split sub-super Lie algebra. Hence, there exists a super Lie group which has $\widehat{g_{Lie}}$ as its super Lie algebra.

- This construction gives a topological foundation to some of the supergroups found in supersymmetric physics.

References

- The original theorems in this talk are a joint work with my advisor Ronald O. Fulp,

  "*INFINITE DIMENSIONAL SUPER LIE GROUPS*"

  math-ph/0610061

- Additional references can be found in our paper.