Chapter 1 Summary:

$$T_p\mathbb{R}^3 = \{ v_p \mid v \in \mathbb{R}^3 \}$$

$\mathfrak{X}(\mathbb{R}^3)$ = all vector fields on $\mathbb{R}^3$, smooth.

$V \in \mathfrak{X}(\mathbb{R}^3) \Rightarrow V[p] \in T_p\mathbb{R}^3$ for each $p \in \mathbb{R}^3$.

(§1.2) • $\mathfrak{X}(\mathbb{R}^3)$ forms module over set of smooth fncts on $\mathbb{R}^3$.
If $V, W \in \mathfrak{X}(\mathbb{R}^3)$ and $f \in C^\infty(\mathbb{R}^3)$ then
$V+W$, $fV \in \mathfrak{X}(\mathbb{R}^3)$ where for each $p \in \mathbb{R}^3$,

$$(V+W)[p] = V[p] + W[p].$$

$$(fV)[p] = f(p)\,V(p).$$

A <u>module</u> is a vector space-like object, difference is the field of scalars is instead a <u>ring</u> of scalars. Smooth fncts form a <u>ring</u>.

$$\mathfrak{X}(\mathbb{R}^3) = \operatorname{span}_{C^\infty}\{U_1, U_2, U_3\}$$

The module of vector fields is a finite-$C^\infty(\mathbb{R}^3)$ module as $V = v_1 U_1 + v_2 U_2 + v_3 U_3 \quad \forall V \in \mathfrak{X}(\mathbb{R}^3)$.

(§1.3) • $$v_p[f] = \underbrace{\frac{d}{dt}\Big[f(p+tv)\Big]\Big|_{t=0}}_{\text{def}^{\underline{n}}} = \underbrace{\sum_{i=1}^{3} v_i \frac{\partial f}{\partial x}(p)}_{\text{Lemma 3.2}} \quad \text{given } v = (v_1, v_2, v_3).$$

directional derivative of f at p in the v-direction.
This measures change in f at p in v-direction.

(§1.3 continued)

$\underline{\text{Th}^{m}}$ (3.3) Let $f, g \in C^\infty(\mathbb{R}^3)$ and $V_p, W_p \in T_p\mathbb{R}^3$, $a, b \in \mathbb{R}$,

(1.) $(aV_p + bW_p)[f] = aV_p[f] + bW_p[f]$

(2.) $V_p[af+bg] = aV_p[f] + bV_p[g]$

(3.) $V_p[fg] = V_p[f]\,g(p) + f(p)\,V_p[g]$

$\underline{\text{Proof}}$: easily follows from $V_p[f] = \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i}(p)$ (Lemma 3.2) where $v = (v_1, v_2, v_3)$

$$
\begin{aligned}
V_p[fg] &= \sum_{i=1}^{3} v_i \frac{\partial}{\partial x_i}(fg) && : \text{Lemma 3.2} \\
&= \sum_{i=1}^{3} v_i \left(\frac{\partial f}{\partial x_i}(p) g(p) + f(p) \frac{\partial g}{\partial x_i}(p) \right) && : \text{product rule for } \tfrac{\partial}{\partial x_i} \\
&= \left(\sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i}(p) \right) g(p) + f(p) \left(\sum_{i=1}^{3} v_i \frac{\partial g}{\partial x_i}(p) \right) && : \text{prop. of } \textstyle\sum_i . \\
&= \underline{V_p[f]\, g(p) + f(p)\, V_p[g]} \; // && (\text{Leibniz Rule})
\end{aligned}
$$

VECTOR FIELDS SHARE SAME PROPERTIES, THEY INHERIT THEM POINT-WISE:

$\underline{\text{Cor}(3.4)}$ Let $V, W \in \mathfrak{X}(\mathbb{R}^3)$, $f, g, h \in C^\infty(\mathbb{R}^3)$

(1.) $(fV + gW)[h] = fV[h] + gW[h]$

(2.) $V[af+bg] = aV[f] + bV[g] \quad \forall a, b \in \mathbb{R}$

(3.) $V[fg] = V[f]\cdot g + f \cdot V[g]$

Notice V takes function $f \in C^\infty(\mathbb{R}^3)$ and produces new function $V[f]$ via the pointwise rule,

$$(V[f])(p) = (V(p))[f]$$

<u>Example</u>:

$$U_1[f] = \frac{\partial f}{\partial x}, \quad U_2[f] = \frac{\partial f}{\partial y}, \quad U_3[f] = \frac{\partial f}{\partial z}$$

$\{U_1, U_2, U_3\}$ are <u>natural</u> <u>coordinate</u> <u>frame</u>.

(§1.4) <u>Curves in $\mathbb{R}^3$</u>

$\alpha \in C^\infty(I, \mathbb{R}^3) \Rightarrow \alpha : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^3$ is a <u>curve</u>

$$\alpha(t) = \big(\alpha_1(t), \alpha_2(t), \alpha_3(t)\big) \in \mathbb{R}^3$$

$$\alpha'(t) = \left(\frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt}, \frac{d\alpha_3}{dt}\right)_{\alpha(t)} = \sum_{i=1}^{3} \frac{d\alpha_i}{dt} U_i \underbrace{(\alpha(t))}_{\text{often omitted.}} \quad ☆☆☆$$

Examples: helix, line, intersection of cylinder & sphere, on $xy=1$, "3-curve"

<u>Concept</u>: a given curve admits many choices of parameter

$\boxed{\text{Def}^{\underline{n}}\ \alpha : I \longrightarrow \mathbb{R}^3 \text{ a curve. If } h : J \longrightarrow I \text{ is diff. on } J \text{ then } \beta = \alpha(h) : J \longrightarrow \mathbb{R}^3 \text{ is a curve called a } \underline{\text{reparametrization}} \text{ of } \alpha}$

(4.5) $\boxed{\underline{\text{Lemma}}: \ \beta'(s) = \frac{d}{ds}\big(\alpha(h(s))\big) = \frac{d\alpha}{ds}(h(s))\,\frac{dh}{ds} = \alpha'(h(s))\,\frac{dh}{ds}}$

(this is how the velocity vectors of a curve & its reparametrization are related.)

$\boxed{\underline{\text{Lemma}}\ (4.6): \ \alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t)}$

<u>Proof</u>:

$$\frac{d}{dt}\Big[f(\alpha(t))\Big] = \frac{\partial f}{\partial x}\frac{d(x(\alpha(t)))}{dt} + \frac{\partial f}{\partial y}\frac{d(y(\alpha(t)))}{dt} + \frac{\partial f}{\partial z}\frac{d(z(\alpha(t)))}{dt}$$

$$= \frac{\partial f}{\partial x}\frac{d\alpha_1}{dt} + \frac{\partial f}{\partial y}\frac{d\alpha_2}{dt} + \frac{\partial f}{\partial z}\frac{d\alpha_3}{dt}$$

$$= \left[\frac{d\alpha_1}{dt}\frac{\partial}{\partial x} + \frac{d\alpha_2}{dt}\frac{\partial}{\partial y} + \frac{d\alpha_3}{dt}\frac{\partial}{\partial z}\right][f]$$

how U_1, U_2, U_3 act on f.

$$= \left(\frac{d\alpha_1}{dt}U_1 + \frac{d\alpha_2}{dt}U_2 + \frac{d\alpha_3}{dt}U_3\right)[f]$$

By ☆☆☆ up page. almost the def$^{\underline{n}}$ of $\alpha'(t)$.

$$= \alpha'(t)[f]$$

<u>Remark</u>: Lemma 4.6 shows rate of change of f along curve α is given by $\alpha'(t)[f]$ ← directional derivative.

(§ 1.5) (one forms and one-form-fields)

Def^{n} (5.1) A 1-form ϕ on $\mathbb{R}^3$ is a real-valued function on the set of all tangent vectors to $\mathbb{R}^3$ such that ϕ is linear at each point, that is:

$$\phi(aV + bW) = a\phi(V) + b\phi(W)$$

for all $a, b \in \mathbb{R}$ and tangent vectors V, W at the same point of $\mathbb{R}^3$

To be more clear, $\phi_p \in (T_p\mathbb{R}^3)^* = \{f : T_p\mathbb{R}^3 \to \mathbb{R} \mid f \text{ linear}\}$

a one-form on $\mathbb{R}^3$ is an assignment $p \mapsto \phi_p$ for each $p \in \mathbb{R}^3$.

Naturally, we could define a one-form along some subset of $\mathbb{R}^3$ just the same.

Def^{n}/ $\Lambda^1(\mathbb{R}^3)$ = set of one forms on $\mathbb{R}^3$, this has a module structure w.r.t. the ring of smooth fncts on $\mathbb{R}^3$. If $f, g \in C^\infty(\mathbb{R}^3)$ and $\phi, \psi \in \Lambda^1(\mathbb{R}^3)$ then

$$(f\phi + g\psi)(p) = f(p)\phi(p) + g(p)\psi(p)$$

When one-forms act on vectors the fncts f, g above basically just ride along, often we use $\psi(p) = \psi_p$ and $\phi(p) = \phi_p$ as shorthand.

$$(f\phi + g\psi)(p)[V_p] = f(p)\,\phi_p(V_p) + g(p)\,\psi_p(V_p)$$

Def^{n}/ (5.2) $df(V_p) = V_p[f]$ ← directional derivative used to define the differential of $f : \mathbb{R}^3 \to \mathbb{R}$

Lemma 5.4: $\phi \in \Lambda^1(\mathbb{R}^3) \Rightarrow \phi = \sum_{i=1}^{3} \phi(U_i)\,dx_i$

I would tend to use ϕ_1, ϕ_2, ϕ_3 as coord. fncts for ϕ where Oneil uses f_1, f_2, f_3. In any event,

Proof: $dx_i(V_p) = V_p[x_i] = V_i$ for $V_p = (V_1, V_2, V_3)_p$.

Calculate, $\phi(V) = \phi(V_1U_1) + \phi(V_2U_2) + \phi(V_3U_3)$ for $V = \sum_{i=1}^{3} V_iU_i$

$$= V_1\phi(U_1) + V_2\phi(U_2) + V_3\phi(U_3)$$

$$= \phi(U_1)\,dx_1(V) + \phi(U_2)\,dx_2(V) + \phi(U_3)\,dx_3(V)$$ the lemma follows. //

(§1.5 summary continued)

$$\boxed{\text{Cor (5.5)} \quad df = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} dx_i}$$

Proof: $df(U_i) = U_i[f] = \frac{\partial f}{\partial x_i} \Rightarrow df = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} dx^i$ by Lemma 5.4.//

Lemma (5.6): Leibniz:

$$d(fg) = (df)g + f\,dg = g\,df + f\,dg$$

denotes point-wise multiplication or scalar multiplication by fncts in the $C^\infty(\mathbb{R}^3)$-module $\Lambda^1(\mathbb{R}^3)$.

Lemma 5.7 (Chain Rule)

$f: \mathbb{R}^3 \to \mathbb{R}$, $h: \mathbb{R} \to \mathbb{R}$ then $d(h(f)) = h'(f)\,df$

Proof:
$$\begin{aligned} d(h\circ f) &= \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(h\circ f)\,dx_i \\ &= \sum_{i=1}^{3} h'(f(x)) \frac{\partial f}{\partial x_i} dx_i \\ &= h'(f(x)) \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} dx_i \\ &= h'(f(x))\,df. // \end{aligned}$$

Chain Rule from calculus III.

Remark: Oneil prefers the notation $h(f)$ for $h\circ f$.

//

(§1.6) - Differential Forms

For $\mathbb{R}^3$ with cartesian coordinates x, y, z the possible types are as follows: f, a, b, c, g all fncts on $\mathbb{R}^3$,

0 - form : f

1 - form : $a\,dx + b\,dy + c\,dz = \omega_{\langle a,b,c\rangle}$ ← "work form"

2 - form : $a\,dy\wedge dz + b\,dz\wedge dx + c\,dx\wedge dy = \Phi_{\langle a,b,c\rangle}$ ← "flux form"

3 - form : $g\,dx\wedge dy\wedge dz$

Note, $dx\wedge dy\wedge dz = \text{Vol}_{\mathbb{R}^3}$ is another common notation as $dx\wedge dy\wedge dz$ is the <u>volume form</u> for $\mathbb{R}^3$. Also, $\omega_{\langle a,b,c\rangle} \neq \Phi_{\langle a,b,c\rangle}$ yet both correspond to vector field $\langle a, b, c\rangle$.

Wedge Product:

Let $dx_i \wedge dx_j = -dx_j \wedge dx_i \quad \forall\, i,j$

then extend this rule associatively, and distributively across the ring of smooth fncts.

$$\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$$

$$\alpha \wedge (c\beta) = c\,\alpha \wedge \beta \quad \text{for any scalar fnct. } c.$$

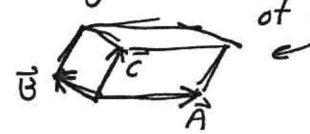
<u>Proposition</u>: If α is p-form & β is q-form then $\alpha \wedge \beta = (-1)^{pq}\, \beta \wedge \alpha$. Thus even degree forms (fncts or two-forms for $\mathbb{R}^3$) commute with all other forms under $\wedge$. On the other hand, 1-forms anticommute (Lemma 6.2)

Most important identities;

$$\omega_{\vec{A}} \wedge \omega_{\vec{B}} = \Phi_{\vec{A}\times\vec{B}}$$

$$\omega_{\vec{A}} \wedge \Phi_{\vec{B}} = \pm(\vec{A}\cdot\vec{B})\, dx\wedge dy\wedge dz$$

$$\omega_A \wedge \omega_B \wedge \omega_C = \pm\underbrace{(\vec{A}\times\vec{B})\cdot\vec{C}}_{\text{triple product gives volume} \pm \text{ of}}\, dx\wedge dy\wedge dz$$



Exterior Derivative:

$$d : \Lambda^p(\mathbb{R}^3) \longrightarrow \Lambda^{p+1}(\mathbb{R}^3)$$

generally, $\alpha \in \Lambda^p(\mathbb{R}^3)$ has form (sorry, pun not intended)

$$\alpha = \sum_{i_1\cdots i_p}{}' \underbrace{\alpha_{i_1\cdots i_p}}_{\text{coeff. fncts. on } \mathbb{R}^3\text{, assumed smooth here.}} dx_{i_1}\wedge dx_{i_2}\wedge\cdots\wedge dx_{i_p}$$

$$d\alpha = \sum_{i_1\cdots i_p}{}' (d\alpha_{i_1\cdots i_p}) \wedge dx_{i_1}\wedge dx_{i_2}\wedge\cdots\wedge dx_{i_p}$$

So, the rule is simply,

"take total differential of coeff. and insert wedge with dx's!"

Case By Case

0-form) $\quad df = \sum_{i=1}^{3}{}' \frac{\partial f}{\partial x_i} dx_i = (\partial_x f)dx + (\partial_y f)dy + (\partial_z f)dz$

1-form) $\quad \alpha = a\,dx + b\,dy + c\,dz$

$$d\alpha = \underline{da \wedge dx + db \wedge dy + dc \wedge dz}$$

I just work this out when a, b, c are explicit.

$$= (\partial_y a\, dy + \partial_z a\, dz)\wedge dx + (\partial_x b\, dx + \partial_z b\, dz)\wedge dy + (\partial_x c\, dx + \partial_y c\, dy)\wedge dz$$

didn't bother to write those $dx\wedge dx$, $dy \wedge dy$, $dz \wedge dz$ coefficients.

$$= \left(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z}\right) dy\wedge dz + \left(\frac{\partial a}{\partial z} - \frac{\partial c}{\partial x}\right) dz \wedge dx + \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx \wedge dy$$

$$= \Phi_{\nabla \times \langle a, b, c\rangle}$$

2-form) $\quad \beta = a\, dy\wedge dz + b\, dz \wedge dx + c\, dx\wedge dy$

$$d\beta = da \wedge dy \wedge dz + db \wedge dz \wedge dx + dc \wedge dx \wedge dy$$

$$= \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}\right) dx \wedge dy \wedge dz$$

$$= (\nabla \cdot \langle a, b, c\rangle)\, dx \wedge dy \wedge dz$$

3-form) $\quad \gamma = g\, dx \wedge dy \wedge dz$

$$d\gamma = dg \wedge dx \wedge dy \wedge dz$$

$$= (\partial_x g\, dx + \partial_y g\, dy + \partial_z g\, dz) \wedge dx \wedge dy \wedge dz$$

$= 0$ as repeated differentials wedge to ZERO.

Properties of d

$d(fg) = (df)g + f\,dg$ for fncts f, g

$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta$

Exampl $\; d(\omega_{\vec A} \wedge \omega_{\vec B}) = d\omega_{\vec A} \wedge \omega_{\vec B} - \omega_{\vec A} \wedge d\omega_{\vec B}$

continuing,

$$d(\omega_{\vec{A}} \wedge \omega_{\vec{B}}) = d\omega_{\vec{A}} \wedge \omega_{\vec{B}} - \omega_{\vec{A}} \wedge d\omega_{\vec{B}}$$

$$\Rightarrow d(\Phi_{\vec{A}\times\vec{B}}) = \Phi_{\nabla\times\vec{A}} \wedge \omega_{\vec{B}} - \omega_{\vec{A}} \wedge \Phi_{\nabla\times\vec{B}}$$

$$\Rightarrow (\nabla\cdot(\vec{A}\times\vec{B}))\,dx\wedge dy\wedge dz = ((\nabla\times\vec{A})\cdot\vec{B} - \vec{A}\cdot(\nabla\times\vec{B}))\,dx\wedge dy\wedge dz$$

Fun fact, $\nabla\cdot(\vec{A}\times\vec{B}) = (\nabla\times\vec{A})\cdot\vec{B} - \vec{A}\cdot(\nabla\times\vec{B})$.

(§1.7) (Mappings) (we'll expand on these points later if needed)

$F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$: is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$

$(F_*)_p : T_p\mathbb{R}^n \longrightarrow T_{F(p)}\mathbb{R}^m$: push-forward, or as Oneil calls it "the tangent map"

I have more to say in 332 notes but, I resist digression here.

Cor 7.6) $(F_*)_p$ is linear

Cor 7.7.) If $\beta = F(\alpha)$ is image of curve α in $\mathbb{R}^n$ to curve β in $\mathbb{R}^m$ then $\beta' = F_*(\alpha')$.

Cor 7.8) $F_*(U_j(p)) = \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_j} \bar{U}_i(F(p))$ for $1 \leq j \leq n$

Jacobian Matrix at p $\left(\frac{\partial f_i}{\partial x_j}(p)\right) = \left[\frac{\partial F}{\partial x_1} \Big| \frac{\partial F}{\partial x_2} \Big| \cdots \Big| \frac{\partial F}{\partial x_n}\right] = J_f^{(p)}$

where $F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$ so $\left(\frac{\partial f_i}{\partial x_j}\right)$ as column index j and row index i.

$\boxed{\text{Defn/ A mapping } F : \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ is \underline{regular} provided at every point } p \text{ of } \mathbb{R}^n \text{ the tangent map } F_{*p} \text{ is 1-1}}$

But, by linear algebra,

$$F_{*p} \text{ is 1-1} \iff F_*(V_p) = 0 \Rightarrow V_p = 0 \iff \text{rank}(J_F^{(p)}) = n.$$

Thm (7.10)) If F is regular at p then $\exists$ a smooth local inverse mapping.//