Notes on Chapter 2 of Oneil

$\boxed{\text{Def}^{n}/ \ p \cdot q = p_1 q_1 + p_2 q_2 + p_3 q_3}$

$$\|p\| = (p \cdot p)^{1/2} = \sqrt{p_1^2 + p_2^2 + p_3^2}$$

$$d(p, q) = \| p - q \|$$

All the fun facts you know and love from calculus III,

$$(ap + bq) \cdot r = ap \cdot r + bq \cdot r$$

$$r \cdot (ap + bq) = ar \cdot p + br \cdot q$$

$$p \cdot p \geq 0 \text{ and } \|p\| = 0 \text{ iff } p = 0.$$

$$\|p + q\| \leq \|p\| + \|q\| \;\; \& \;\; \|cp\| = |c| \, \|p\|$$

We define dot product on $T_p\mathbb{R}^3$ by using the natural dot-product transferred by $V_p \longleftrightarrow V$

$\boxed{\text{Def}^{n}/ \ V_p \cdot W_p = V \cdot W}$

V, W are orthogonal iff $V \cdot W = 0$. Also, we define $\theta \in [0, \pi]$ by $V \cdot W = \|V\| \|W\| \cos\theta \rightarrow \underline{V \perp W \text{ if } \theta = \frac{\pi}{2}}$.

$\boxed{\text{Def}^{n}/ \ e_1, e_2, e_3 \text{ is a frame at } p \in \mathbb{R}^3 \text{ if } e_1, e_2, e_3 \in T_p\mathbb{R}^3 \text{ and } e_i \cdot e_j = \delta_{ij} \ \forall i, j.}$

<u>Example</u>: $U_1(p), U_2(p), U_3(p)$ form a frame at p.

$\boxed{\text{Th}^{m}/ \text{ Let } e_1, e_2, e_3 \text{ be a frame for } T_p\mathbb{R}^3 \text{ then if } v \in T_p\mathbb{R}^3 \text{ then } v = (v \cdot e_1)e_1 + (v \cdot e_2)e_2 + (v \cdot e_3)e_3}$

<u>Proof</u>: Since $\{e_1, e_2, e_3\}$ is orthogonal $\Rightarrow \{e_1, e_2, e_3\}$ LI hence $\exists \, c_1, c_2, c_3$ for which $v = c_1 e_1 + c_2 e_2 + c_3 e_3$. Now take dot-products with e_1, e_2, e_3 to conclude. //

The Euclidean Frame U_1, U_2, U_3 gives

$$V_p = (v_1, v_2, v_3)_p = \sum_{i=1}^{3} v_i U_i(p)$$

Another frame,

$$V_p = \sum_{i=1}^{3} a_i e_i$$

Likewise for W_p. Note,

$$\begin{aligned}
V_p \cdot W_p &= \Big(\sum_i a_i e_i\Big)\cdot\Big(\sum_j b_j e_j\Big) \\
&= \sum_{i,j} a_i b_j \, e_i \cdot e_j \\
&= \sum_{i,j} a_i b_j \delta_{ij} \\
&= \underbrace{a_1 b_1 + a_2 b_2 + a_3 b_3}_{\{e_i\}\text{ frame formula.}} = \underbrace{v_1 w_1 + v_2 w_2 + v_3 w_3}_{\text{Euclidean Coordinates formula}}
\end{aligned}$$

Def^{n}/ Let e_1, e_2, e_3 be frame at $p \in \mathbb{R}^3$ then the <u>attitude</u> <u>matrix</u> of the frame is (a_{ij}) given by the following:

$$\begin{aligned}
e_1 &= (a_{11}, a_{12}, a_{13})_p \\
e_2 &= (a_{21}, a_{22}, a_{23})_p \\
e_3 &= (a_{31}, a_{32}, a_{33})_p
\end{aligned}
\longrightarrow \underbrace{A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\text{attitude matrix.}}$$

<u>Observe</u>: $\displaystyle\sum_{k=1}^{3} a_{ik} \underbrace{a_{jk}}_{(a^T)_{kj}} = a_i \cdot a_j = \delta_{ij}$

$$\longrightarrow \underbrace{AA^T = I}_{\text{or } A^tA = I \text{ for Oneil.}}$$

Def^{n}/ $V \times W = \begin{vmatrix} U_1(P) & U_2(P) & U_3(P) \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{vmatrix}$

<u>Lemma</u> (1.8) $\|V \times W\|^2 = (V \cdot V)(W \cdot W) - (V \cdot W)^2$

<u>Proof</u>: see my calculus III notes or pg. 49.
This is Lagranges' Identity, it's non trivial. //

<u>Cor</u>: $\|V \times W\| = \|V\| \|W\| \sin\theta$ and R.H.R. gives direction.

<u>Exercises on p. 50</u> (calculus III in new notation)

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<u>§2.2 Curves</u>

$\alpha : I \longrightarrow \mathbb{R}^3$ a curve

$$\alpha'(t) = \left(\frac{d\alpha_1}{dt}(t), \frac{d\alpha_2}{dt}(t), \frac{d\alpha_3}{dt}(t) \right)_{\alpha(t)}$$

<u>speed</u> $V = \|\alpha'\| = \sqrt{\alpha'(t) \cdot \alpha'(t)}$

arclength from $\alpha(a)$ to $\alpha(b)$ $= \int_a^b \|\alpha'(t)\| \, dt$

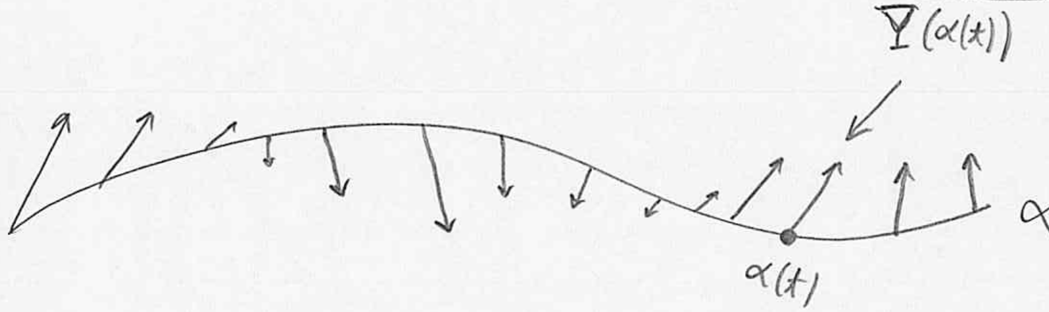
Th^{m}/ If α a regular curve in $\mathbb{R}^3$ then $\exists$ a reparametrization β such that β has unit speed

<u>Proof</u>: parametrize by arclength. Then

$$\|\beta'(s)\| = \frac{dt}{ds} \|\alpha'(t(s))\| = \frac{dt}{ds}\frac{ds}{dt} = 1. //$$

(I have careful proof in 231 notes or see pg. 53)

Def^n/ A vector field Y on curve $\alpha : I \to \mathbb{R}^3$ is a function that assigns to each $t \in I$ a tangent vector $Y(t)$ at $\alpha(t)$.



To differentiate Y above, we differentiate the component fncts. If $Y = \sum y_i U_i$ then

$$Y' = \frac{dY}{dt} = \sum_{i=1}^{3} \frac{dy_i}{dt} U_i$$

Likewise Y'' is defined.

PROPERTIES

$$(aY + bZ)' = aY' + bZ'$$

$$(fY)' = \frac{df}{dt} Y + f Y'$$

$$\boxed{(Y \cdot Z)' = Y' \cdot Z + Y \cdot Z'}$$

THIS IS IMPORTANT!

Lemma 2.3:

(1) curve α is constant iff $\alpha' = 0$
(2) a non constant curve α is straight line iff $\alpha'' = 0$
(3.) a vector field Y on a curve is parallel iff $Y' = 0$

Proof: see text.//   Exercises pg. 57 (11 problems)

Proof of Lemma 2.3

(1.) a curve α is constant iff $\alpha' = 0$

Proof: If $\alpha(t) = p \;\; \forall t$ then $\alpha'(t) = 0 \Rightarrow \alpha' = 0$.
Conversely $\alpha' = \left(\frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt}, \frac{d\alpha_3}{dt}\right) = (0,0,0)$ at each $(\alpha(t))$
thus $\frac{d\alpha_i}{dt} = 0 \Rightarrow \alpha_i = c_i \Rightarrow \alpha(t) = (c_1, c_2, c_3)$. //

(2.) a non-constant curve α is straight line iff $\alpha'' = 0$.

If α straight line then $\exists\, p, v$ s.t $\alpha(t) = p + tv$ thus
$\alpha'(t) = v_{\alpha(t)} = (v_1, v_2, v_3) \Rightarrow \alpha''(t) = \left(\frac{d}{dt}(v_1), \frac{d}{dt}(v_2), \frac{d}{dt}(v_3)\right) = (0,0,0)$
thus $\alpha'' = 0$. Conversely if $\alpha'' = 0$ then
$\alpha = (\alpha_1, \alpha_2, \alpha_3)$ has $\alpha'' = (\alpha_1'', \alpha_2'', \alpha_3'') = (0,0,0)$
So integrate twice to find $\alpha(t) = (p_1 + tv_1, p_2 + tv_2, p_3 + tv_3)$
which is a straight line. //

$\left\{ \right.$ Remark: soon after, β unit speed has $\beta' = T$
and $\beta'' = \kappa N$ hence $\beta'' = 0 \Rightarrow \kappa = 0$.
we see straight lines have zero curvature. $\left. \right\}$

(3.) a vector field Y on a curve α is parallel iff $Y' = 0$.

$\Rightarrow$) If $Y \parallel \alpha$ then $Y(t) = (c_1, c_2, c_3)_{\alpha(t)} \;\; \forall t$
$\Rightarrow Y'(t) = \left(\frac{d}{dt}(c_1), \frac{d}{dt}(c_2), \frac{d}{dt}(c_3)\right) = (0,0,0)$.

$\Leftarrow$) If $\frac{dY}{dt} = (0,0,0)$ then $\Rightarrow \left(\frac{dY_1}{dt}, \frac{dY_2}{dt}, \frac{dY_3}{dt}\right) = (0,0,0)$
$\Rightarrow Y(t) = (c_1, c_2, c_3)_{(\alpha(t))}$ //

§ 2.3 THE FRENET FORMULAS

$\beta : I \longrightarrow \mathbb{R}^3$ unit speed , $\|\beta'(s)\| = 1$.

$\boxed{\text{Def}^{n} \quad T = \beta' \text{ is unit tangent vector field,}}$

$T' = \beta''$ measures change in direction for T as $T \cdot T = \|T\|^2 = 1$.

curvature vector field for β

Note $T \cdot T = 1 \Rightarrow 2T' \cdot T = 0$

$\Rightarrow T' \perp T$

$\boxed{\text{Def}^{n} / \ \kappa(s) = \|T'(s)\| = \text{curvature fnct. of } \beta}$

Def^n/ $N = T'/\kappa$ and $B = T \times N$

unit normal — binormal

Lemma 3.1: β a unit speed curve with $\kappa > 0$. The three vector fields T, N, B are unit-vector fields which are mutually $\perp$. These form the Frenet Frame for β

(I begin again and work out Frenet Serret like Oneil for next two pages, we define κ and T again as we go)

Frenet Serret Derivation

Btw this is unique up to base point $\Rightarrow \kappa$ and τ well-defined.

Let α be a unit-speed curve; $\alpha'(s) \cdot \alpha'(s) = 1$ for all s. We define $\boxed{T = \alpha'}$, the unit-tangent to α (TANGENT)

Next, differerentiate $\alpha' \cdot \alpha' = 1$ and use product rule,

$$\alpha'' \cdot \alpha' + \alpha' \cdot \alpha'' = 0$$

$$\Rightarrow \alpha'' \cdot T = 0$$

The vector field α'' decribes the change in T, it thus describes how T is turning given $\|T\| = 1$

We define $\kappa = \|\alpha''\| = \sqrt{\alpha'' \cdot \alpha''}$. If $\kappa > 0$ (assumed) then define $\boxed{N = \frac{1}{\kappa}\alpha'' = \frac{1}{\kappa}T'}$. Observe $N \cdot T = 0$ (NORMAL)

as $\frac{1}{\kappa}\alpha'' \cdot T = \frac{1}{\kappa}(\alpha'' \cdot T) = 0$. Also $N \cdot N = 1$

To complete the Frenet Frame define $\underbrace{\boxed{B = T \times N}}_{\text{BINORMAL}}$

The Frenet - Serret Eq$^{\text{n}}$'s detail the change in T, N, B in terms of T, N, B. Observe, we already have

1.) $\boxed{T' = \kappa N}$

Notice, $\{T, N, B\}$ is orthonormal as $T \cdot T = N \cdot N = 1$ and $T \cdot N = 0 \Rightarrow B \cdot B = \|T \times N\| = \sin\theta = 1$. (using Lagrange's Identity which motivates $T \times N = (TN \sin\theta)\hat{n}$. Furthermore, as $T \cdot (T \times N) = N \cdot (T \times N) = 0$ we have $T \cdot B = N \cdot B = 0$. Therefore $\{T, N, B\}$ is orthonormal frame as claimed.

$$B' = (B' \cdot T)T + (B' \cdot N)N + (B' \cdot B)B \quad \text{as } B' \in \text{span}\{T, N, B\}.$$

Note, $B \cdot B = 1 \Rightarrow B' \cdot B + B \cdot B' = 0 \Rightarrow \underline{B' \cdot B = 0}$.

Also, $B \cdot T = 0 \Rightarrow B' \cdot T + B \cdot T' = 0 \Rightarrow B' \cdot T + \kappa \cancel{B \cdot N}^{\,0} = 0$

Thus $B' \cdot T = 0$. We find $B' = (B' \cdot N)N$ so define $\boxed{\tau = -B' \cdot N}$

2.) $\boxed{B' = -\tau N}$

continued

Both $T' = \kappa N$ and $B' = -\tau N$ are essentially the definitions of curvature κ and torsion τ. The curvature measures how T bends along curve whereas the torsion τ measures how B bends along curve. Since we define N to be the direction in which T changes this ultimately forces $B = T \times N$ to also change in the N direction (if $\tau \neq 0$). Finally we consider N, I suppose it's not surprising it changes into both T & B directions,

$$N' = (N' \cdot T)T + (N' \cdot N)N + (N' \cdot B)B$$

Note $N \cdot N = 1 \Rightarrow 2N' \cdot N = 0$.

Note $N \cdot T = 0 \Rightarrow N' \cdot T + N \cdot T' = 0 \Rightarrow N' \cdot T = -N \cdot (\kappa N) = \underline{-\kappa}$

Also, $N \cdot B = 0 \Rightarrow N' \cdot B + N \cdot B' = 0 \Rightarrow N' \cdot B = -N \cdot B' = -(-\tau) = \underline{\tau}$

Therefore,

$$3.)\ \boxed{N' = -\kappa T + \tau B}$$

<u>Reminder</u>: Eq^n's 1, 2, 3 are all for unit-speed curves with $\kappa > 0$. So, explicitly,

$$T' = \frac{dT}{ds}, \quad N' = \frac{dN}{ds}, \quad B' = \frac{dB}{ds}$$

To reparametrize with a different parameter we change $s \longmapsto h(t)$ and use $\tilde{F} = (F \circ h)$

$$\frac{dF}{ds} = \frac{dt}{ds}\frac{d\tilde{F}}{dt} = \frac{1}{v}\frac{d\tilde{F}}{dt}$$

Then multiply by v to obtain (using $\tilde{\kappa} = \kappa \circ h$, $\tilde{\tau} = \tau \circ h$,

$$\frac{d\tilde{T}}{dt} = \tilde{\kappa} v \tilde{N}, \quad \frac{d\tilde{N}}{dt} = -\tilde{\kappa} v \tilde{T} + \tau v \tilde{B}, \quad \frac{d\tilde{B}}{dt} = -v\tilde{\tau}\tilde{N}$$

Of course, Oneil uses a different notation scheme to explain this in §2.4.

Cor 3.5

Let β be a unit speed curve in $\mathbb{R}^3$ with $\kappa > 0$. Then β is planar curve iff $\tau = 0$.

Lemma 3.6.

If β is a unit-speed curve with constant curvature $\kappa > 0$ and zero torsion, then β is part of a circle of radius $1/\kappa$.

Discussion: Σ a sphere, β unit-speed curve
minimum curvature occurs on great circles (have radius a)
so $\kappa \geq \frac{1}{a}$ (see pg. 66 in r. 2nd Ed)

§2.4 (Arbitrary Speed Curves)

$$\alpha(t) = \bar{\alpha}(s(t)) \quad \text{for all } t$$

$$\kappa = \bar{\kappa}(s)$$
$$\tau = \bar{\tau}(s)$$
$$T = \bar{T}(s)$$
$$N = \bar{N}(s)$$
$$B = \bar{B}(s)$$

(p 70)

Modified Frenet Serret: (just add chainrule past the derivation we already went through on pg. 60

$$T' = \kappa v N$$
$$N' = -\kappa v T + \tau v B$$
$$B' = -\tau v N$$

Lemma 4.2: If α is regular curve with speed v then velocity α' and acceleration α'' are

$$\alpha' = vT, \quad \alpha'' = \frac{dv}{dt} T + \kappa v^2 N$$

Th^m (4.3) gives convenient f-la's for explicit calculation.

Def^n (4.5) A regular curve α in $\mathbb{R}^3$ is a <u>cylindrical helix</u> provided unit tangent T of α has constant angle ϑ with some fixed unit-vector u; $T(t) \cdot u = \cos\vartheta$ $\forall t$.

Th^m (4.6) A regular curve α with $\kappa > 0$ is cylindrical helix iff τ/κ is constant.

Punchline:

$\kappa = 0 \iff$ straight line
$\tau = 0 \iff$ plane curve
$\kappa \equiv$ const > 0, $\tau = 0 \iff$ circle
κ const > 0, τ const $> 0 \iff$ circular helix
τ/κ const $\neq 0 \iff$ cylindrical helix.

κ & τ characterize the curve.