§2.5 COVARIANT DERIVATIVES

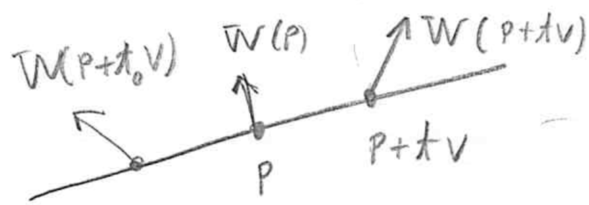
We defined the derivative of $Y \in \mathfrak{X}(\alpha)$ = vector field along α with respect to the parameter of α; $\frac{dY}{dt}$. Personally, I'd like to see some manifest α-dependence like $\frac{dY_\alpha}{dt}$ or something ... but, the α is implicit.

Now we define a new differentiation

> Def^{n}/ Let $W \in \mathfrak{X}(\mathbb{R}^3)$ and $v \in T_p\mathbb{R}^3$. Then the covariant derivative of W w.r.t. v is the tangent vector $\nabla_v W = W(p+tv)'(0)$ at p

Let's unravel this formula,

- $t \mapsto p + tv$ is line through p with direction v
- $W(p+tv)$ gives vector output of $W \in \mathfrak{X}(\mathbb{R}^3)$ at the point $p+tv$.



- $W(p+tv)'(0)$ gives vector in which the change of W point as we go in v-direction at p. Or as O'neil says,

"$\nabla_v W$ measures initial rate of change of $W(p)$ as p moves in v-direction"

This would be a bit more natural with $(\nabla_v W)(p)$. Or, perhaps, $\nabla_{v_p} W$ as $v_p \in T_p\mathbb{R}^3$ anyways.

1. <u>Common sense</u>: If $W = aU_1 + bU_2 + cU_3$ for constants $a, b, c \in \mathbb{R}$ then $\nabla_{V_p} W = 0$ as W does not change. Check,

$$W(P+tV) = (a,b,c)_P$$
$$\Rightarrow \big(W(P+tV)\big)'(t) = (0,0,0).$$

2. <u>Example</u>: Let $W = x^2 U_1 + U_2 + U_3$. Consider V_p with $V = U_1$ then

$$W(P+tV) = W(P_1+t, P_2, P_3)$$
$$\Rightarrow W(P+tV) = (P_1+t)^2 U_1 + U_2 + U_3$$
$$\frac{d}{dt}\big(W(P+tV)\big) = 2(P_1+t)U_1$$

$$\therefore \boxed{\nabla_{V_p} W = 2P_1 U_1}$$

W increases in the U_1 direction as P goes in U_1 directi.

Next, V_p with $V = (0,a,b)$

$$P + tV = (P_1, P_2+ta, P_3+tb)$$
$$W(P+tV) = \underbrace{P_1^2 U_1 + U_2 + U_3}_{\text{constant in } t.}$$

the definition of W forbids any y, z - dep.

$$\therefore \boxed{\nabla_{V_p} W = 0}$$

3. <u>Example</u>: $W = U_1 + x^2 U_2$. If $V = (a,b,c)$ then $P+tV = (P_1+at, P_2+bt, P_3+ct)$ thus, $W(P+tV) = U_1 + (P_1+at)^2 U_2$. Hence, diff. w.r.t. t,

$$\frac{d}{dt}\big[W(P+tV)\big] = 2a(P_1+at)U_2 \xrightarrow{t=0} \boxed{\nabla_{(a,b,c)_p} W = 2aP_1 U_2}$$

W increases in the U_2-direction if we go in U_1-direction.

Calculation in style of last page is tiresome.
We use $\frac{d}{dt}\Big|_{t=0}$ to frame definitions, but eventually, we prefer coordinate calculus!

$$\boxed{\textbf{Lemma (5.2)} \quad \text{If } W = \sum_i w_i U_i \in \mathfrak{X}(\mathbb{R}^3) \text{ and } V \in T_p\mathbb{R}^3 \text{ then } \nabla_{V_p} W = \sum_i V[W_i]\, U_i(p)}$$

Proof:
$$\frac{d}{dt}\Big(W(p+tV)\Big) = \frac{d}{dt}\left[\sum_{i=1}^{3} W_i(p+tV)\, U_i\right]$$
$$= \sum_{i=1}^{3} \frac{d}{dt}\left[W_i(p+tV)\right] U_i$$

chain rule. $W_i = W_i(x_1, x_2, x_3)$

$$= \sum_{i=1}^{3}\left(\sum_{j=1}^{3} \frac{\partial W_i}{\partial x_j}\frac{d(p_j + V_j t)}{dt}\right) U_i$$
$$= \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial W_i}{\partial x_j} V_j\, U_i$$
$$= \sum_{i=1}^{3} \underbrace{\left(\sum_{j=1}^{3} V_j \frac{\partial}{\partial x_j}\right)(W_i)}_{\frac{\partial W_i}{\partial x_j} \text{ evaluated at } p+tV} U_i$$

Thus,
$$\frac{d}{dt}\Big(W(p+tV)\Big)\Big|_{t=0} = \sum_{i=1}^{3}\left(\sum_{j=1}^{3} V_j \frac{\partial}{\partial x_j}\Big|_p\right)(W_i)\, U_i$$
$$\therefore \quad \nabla_{V_p} W = \sum_{i=1}^{3} V_p[W_i]\, U_i$$

Example: $W = U_1 + x^2 U_2$ and $V_p = (a, b, c)_p$

$$\nabla_{V_p} W = \cancel{V_p[1]}\overset{0}{U_1} + V_p[x^2]\, U_2 + \cancel{V_p[0]}\overset{0}{U_3}$$
$$= (aU_1 + bU_2 + cU_3)[x^2]\, U_2$$
$$= 2xa\, U_2 \quad \text{at } x = p_1$$
$$= \underline{2p_1 a\, U_2}.$$

(which is what we found from def^{n} on last page)

Comment: Since $V_p[W_i] = dW_i(V_p)$ we could also write $\nabla_{V_p} W = \sum_{i=1}^{3} V_p[W_i]\, U_i = \sum_{i=1}^{3} dW_i(V_p)\, U_i$.

Example: $W = f U_1 + g U_2$ where f, g are fncts of x, y, z

$W_1 = f$, $W_2 = g$, $W_3 = 0$ hence,

$$\nabla_{V_p} W = df(V_p) U_1 + dg(V_p) U_2 \qquad V_p = (a,b,c)_p$$

$$= \underbrace{\left(a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + c\frac{\partial f}{\partial z}\right)}U_1 + \underbrace{\left(a\frac{\partial g}{\partial x} + b\frac{\partial g}{\partial y} + c\frac{\partial g}{\partial z}\right)}U_2$$

evaluated at p.

Silly examples aside, the theorem to follow is useful,

Th$^{\underline{m}}$ (5.3) PROPERTIES OF COVARIANT DERIVATIVE (at a Point p)

Let $V_p, W_p \in T_p\mathbb{R}^3$ and $Y, Z \in \mathfrak{X}(\mathbb{R}^3)$ then for $a, b \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R}^3)$,

(1) $\nabla_{aV_p + bW_p} Y = a\nabla_{V_p} Y + b\nabla_{W_p} Y$

(2.) $\nabla_{V_p}(aY + bZ) = a\nabla_{V_p} Y + b\nabla_{V_p} Z$

(3.) $\nabla_{V_p}(fY) = V_p[f]\, Y(p) + f(p)\nabla_{V_p} Y$

(4.) $V_p[Y \cdot Z] = (\nabla_{V_p} Y)\cdot Z(p) + Y(p)\cdot(\nabla_{V_p} Z)$

Proof on next page.

Naturally, we can do the same for $V_p \longmapsto V$ as follows, $(\nabla_V W)(p) = \nabla_{V(p)} W$. I put V_p and W_p to emphasize this

Corollary (5.4) PROPERTIES OF COVARIANT DERIVATIVE ~~FOR~~ ALONG VECTOR FIELDS

Let $V, W, Y, Z \in \mathfrak{X}(\mathbb{R}^3)$ and f, g functions from $\mathbb{R}^3$ to $\mathbb{R}$,

(1) $\nabla_{fV + gW} Y = f\nabla_V Y + g\nabla_W Y$

(2.) $\nabla_V(aY + bZ) = a\nabla_V Y + b\nabla_V Z$ for $a, b \in \mathbb{R}$.

(3.) $\nabla_V(fY) = V[f]\, Y + f\nabla_V Y$

(4.) $V[Y \cdot Z] = (\nabla_V Y)\cdot Z + Y \cdot (\nabla_V Z)$.

Proof: (3.) $(\nabla_V(fY))(p) = \nabla_{V(p)}(fY)$

(this is what O'neil omits)

$= V(p)[f]\, Y(p) + f(p)\nabla_{V(p)} Y$ — By (3.) of Th$^{\underline{m}}$ 5.3

$= (V[f]\, Y + f\nabla_V Y)(p)$ likewise for 1, 2, 4. //

Proof of (1) of 5.3 : Let $V_p = (v_1, v_2, v_3)_p$, $W_p = (w_1, w_2, w_3)_p$, $Y = y_1 U_1 + y_2 U_2 + y_3 U_3$,

$$\nabla_{aV_p + bW_p} Y = \sum_{i=1}^{3} (aV_p + bW_p)[y_i]\, U_i$$
$$= a\Big(\sum_i V_p[y_i]\, U_i\Big) + b\Big(\sum_i W_p[y_i]\, U_i\Big)$$
$$= \underline{a \nabla_{V_p} Y + b \nabla_{W_p} Y} \quad //$$

Proof of (3) of 5.3: Same notation once more, f a fnct,

$$\nabla_{V_p}(fY) = \sum_i V_p(fy_i)\, U_i \qquad : \text{noted } fY = fy_1 U_1 + fy_2 U_2 + fy_3 U_3$$
$$= \sum_i \big(V_p[f]\, y_i + f(p)\, V_p[y_i]\big) U_i$$
$$= V_p[f] \underbrace{\sum_i y_i(p)\, U_i}_{Y(p)} + f(p) \underbrace{\sum_i V_p[y_i]\, U_i}_{\nabla_{V_p} Y} = \underline{V_p[f]\, Y(p) + f(p)\, \nabla_{V_p} Y} \quad //$$

Proof of (4) of 5.3

$$V_p[Y \cdot Z] = V_p\Big[\sum_i y_i z_i\Big]$$
$$= \sum_i V_p[y_i z_i]$$
$$= \sum_i \big(V_p[y_i]\, z_i(p) + y_i(p)\, V_p[z_i]\big)$$
$$= \underline{(\nabla_{V_p} Y) \cdot Z(p) + Y(p) \cdot \nabla_{V_p} Z}.$$

noted the i^{th} component of $\nabla_{V_p} Y$ is coeff. of U_i which is simply $V_p[y_i]$.

Now, a pointless example

Let $V = (x^2+y^2) U_1 + yz U_3$ and $W = x U_1 + y^2 U_2 + z^3 U_3$

$$\nabla_V W = \sum_i V[w_i]\, U_i = V[x] U_1 + V[y^2] U_2 + V[z^3] U_3$$
$$= (x^2+y^2) U_1 + 3z^2 yz\, U_3$$
$$= \underline{(x^2+y^2) U_1 + 3yz^3 U_3}.$$

If ~~Bo~~ $Z = y^2 U_1 + x^2 U_2 + z^2 U_3$ then,

$$\nabla_V Z = \cancel{V[y^2]}^{\,0}\, U_1 + V[x^2] U_2 + V[z^2] U_3$$
$$= \underline{(x^2+y^2)(2x) U_2 + 2yz^2 U_3}.$$

like O'neil's Example on pg. 83.

$$\nabla_V W = \sum_{i=1}^{3} V[w_i]\, U_i$$

$$\nabla_{U_1} W = \sum_{i=1}^{3} U_1[w_i]\, U_i = \sum_{i=1}^{3} \frac{\partial w_i}{\partial x} U_i = \frac{\partial W}{\partial x}$$

$$\nabla_{U_2} W = \sum_{i=1}^{3} \frac{\partial w_i}{\partial y} U_i = \frac{\partial W}{\partial y}$$

$$\nabla_{U_3} W = \sum_{i=1}^{3} \frac{\partial w_i}{\partial z} U_i = \frac{\partial W}{\partial z}$$

oh, these were not defined, but, seems natural.

§2.6 FRAME FIELDS

$\boxed{\text{Def}^{\underline{n}}/\ E_1, E_2, E_3 \in \mathcal{X}(\mathbb{R}^3) \text{ are a frame field on } \mathbb{R}^3 \text{ iff } E_i \cdot E_j = \delta_{ij}}$

This simply means $E_1(P)$, $E_2(P)$, $E_3(P)$ form an orthonormal basis of $T_P\mathbb{R}^3$ at each $P \in \mathbb{R}^3$.

Ex/ U_1, U_2, U_3 form a frame field.

Ex/ If $R^T R = I$ then $R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$ gives

$$E_1 = \sum_{j=1}^{3} R_{j1} U_j, \quad E_2 = \sum_{j=1}^{3} R_{j2} U_j \quad \text{and} \quad E_3 = \sum_{j=1}^{3} R_{j3} U_j$$

another constant frame for $\mathbb{R}^3$ (it's just a rotation of the standard Euclidean frame)

$$E_i \cdot E_j = \sum_k \sum_l R_{ki} R_{lj} \underbrace{U_k \cdot U_l}_{\delta_{kl}} = \sum_k (R^T)_{ik} R_{kj} = I_{ij} = \delta_{ij}.$$

In calculus III I champion use of non cartesian coordinate systems. My notation is $\hat{u} = \frac{\nabla u}{\|\nabla u\|}$ ~ points in direction of increasing u - coordinate. For example,

$$\hat{x} = \frac{\nabla x}{\|\nabla x\|} = \langle 1, 0, 0 \rangle, \quad \hat{y} = \langle 0, 1, 0 \rangle, \quad \hat{z} = \langle 0, 0, 1 \rangle$$

Of course $\hat{i}, \hat{j}, \hat{k}$ are typically used for $\hat{x}, \hat{y}, \hat{z}$ in multivariate calculus. Moreover, we "move $\hat{i}, \hat{j}, \hat{k}$ around" so there is implicit use of $T_p\mathbb{R}^3$, I mean, we do think about flux of $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ and $\vec{F}(P_0) = (P, Q, R)_{P_0}$ in our language.

I usually introduce formulas for gradient, curl and divergence in polar/cylindrical and spherical coordinates. To accomplish that goal I need analogs of $\hat{x}, \hat{y}, \hat{z}$ for r, θ, z and ρ, ϕ, θ these are simply $\hat{r}, \hat{\theta}, \hat{z}$ and $\hat{\rho}, \hat{\phi}, \hat{\theta}$. I have dozens of pages devoted to their derivation. I'll probably show you my visualizations of $\hat{\rho}, \hat{\phi}, \hat{\theta}$ via MAPLE.

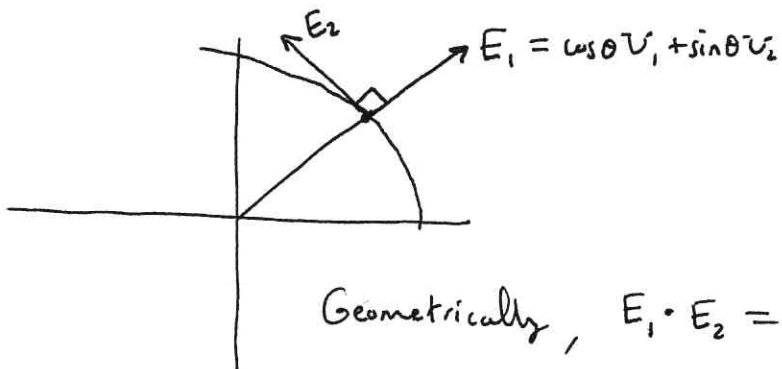
Example:

$x = r\cos\theta \longrightarrow r^2 = x^2 + y^2 \rightarrow 2r\,dr = 2x\,dx + 2y\,dy$

$y = r\sin\theta \longrightarrow \tan\theta = y/x$

$z = z$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$



$$\nabla r = \left\langle \frac{x}{r}, \frac{y}{r}, 0 \right\rangle$$

$$\Rightarrow \underline{\nabla r = \cos\theta\, U_1 + \sin\theta\, U_2 = E_1}$$

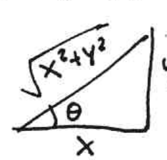
Geometrically, $E_1 \cdot E_2 = 0$ and $\|E_2\| = 1$

$$\Rightarrow \underline{E_2 = -\sin\theta\, U_1 + \cos\theta\, U_2}.$$

Of course, $E_3 = U_3$ so $\{E_1, E_2, E_3\}$ forms cylindrical coord. frame.

(or just calculate it, $\tan\theta = y/x \Rightarrow \sec^2\theta\, d\theta = \dfrac{dy}{x} - \dfrac{y\,dx}{x^2}$

and



$\Rightarrow \sec\theta = \dfrac{\sqrt{x^2+y^2}}{x} \quad \therefore \quad \dfrac{(x^2+y^2)\,d\theta}{x^2} = \dfrac{x\,dy - y\,dx}{x^2}$

$$\therefore d\theta = \frac{x\,dy - y\,dx}{x^2+y^2} = \omega_{\nabla\theta}$$

$$\nabla\theta = \left\langle \frac{-\sin\theta}{r}, \frac{\cos\theta}{r} \right\rangle$$

$$\Rightarrow E_2 = \langle -\sin\theta, \cos\theta, 0 \rangle.$$

FRAME FIELDS CONTINUED

(Note: this has little to do with our main aim in § 2.5 – 2.8, can ignore details here!)

Ex] $x = \rho\cos\theta\sin\phi$
$y = \rho\sin\theta\sin\phi$
$z = \rho\cos\phi$

$$x^2+y^2+z^2 = \rho^2 \Rightarrow 2x\,dx + 2y\,dy + 2z\,dz = 2\rho\,d\rho$$

$$\Rightarrow d\rho = \frac{1}{\rho}(x\,dx + y\,dy + z\,dz) = \omega_{\nabla\rho}$$

$$\therefore \nabla\rho = \left\langle \frac{x}{\rho}, \frac{y}{\rho}, \frac{z}{\rho} \right\rangle$$

$$\Rightarrow \underline{E_1 = \cos\theta\sin\phi\, U_1 + \sin\theta\sin\phi\, U_2 + \cos\phi\, U_3}.$$

$\tan\theta = \frac{y}{x}$
same as cylindricals $\Rightarrow$ $\underline{E_3 = -\sin\theta\, U_1 + \cos\theta\, U_2}$.

Finally $\cos\phi = \frac{z}{\rho}$ $\Rightarrow$ $-\sin\phi\, d\phi = \frac{dz}{\rho} - \frac{z\,d\rho}{\rho^2}$

$$\Rightarrow -\sin\phi\, d\phi = \frac{1}{\rho}\left(dz - \frac{z}{\rho}\frac{1}{\rho}(x\,dx + y\,dy + z\,dz)\right)$$

$$\Rightarrow -\rho\sin\phi\, d\phi = \frac{-xz}{\rho^2}dx - \frac{yz}{\rho^2}dy + \left(1 - \frac{z^2}{\rho^2}\right)dz$$

$$\Rightarrow -\rho\sin\phi\, d\phi = \frac{-xz\,dx - yz\,dy + (x^2+y^2)\,dz}{\rho^2}$$

$$\Rightarrow \rho\sin\phi\, d\phi = \cos\theta\sin\phi\cos\phi\,dx + \sin\phi\sin\theta\cos\phi\,dy + \sin^2\phi\,dz$$

$$\Rightarrow d\phi = \frac{1}{\rho}\left(\cos\theta\cos\phi\,dx + \sin\theta\cos\phi\,dy + \sin\phi\,dz\right)$$

$$\overset{\shortparallel}{\omega_{\nabla\phi}} \Rightarrow \nabla\phi = \frac{1}{\rho}\Big[\underbrace{\cos\theta\cos\phi\, U_1 + \sin\theta\cos\phi\, U_2 + \sin\phi\, U_3}_{\text{unit-vector}}\Big]$$

this is E_2.

I number them as I did so that $E_1 \times E_2 = E_3$
we learn this in Chapter 3.
$\{E_1, E_2, E_3\}$ is right-handed-frame.

Lemma (6.3) Let E_1, E_2, E_3 be a frame field on $\mathbb{R}^3$.

(1.) If $V \in \mathfrak{X}(\mathbb{R}^3)$ then $V = \sum_i f_i E_i$ where $f_i = V \cdot E_i$ or more compactly, $V = \sum_i (V \cdot E_i) E_i$ so the component fncts of V in the E_1, E_2, E_3 frame are obtained through dot-products with E_1, E_2, E_3 respective.

(2.) If $V, W \in \mathfrak{X}(\mathbb{R}^3)$ and $V = \sum f_i E_i$ & $W = \sum_i g_i E_i$ then $V \cdot W = \sum_i f_i g_i$. Moreover, $\|V\| = \sqrt{\sum_i f_i^2}$

Proof: Let $V = \sum_{i=1}^{3} V_i U_i$ and $W = \sum_{i=1}^{3} W_i U_i$

(1) We're given $E_i \cdot E_j = \delta_{ij}$ hence $\{E_1, E_2, E_3\}$ forms basis for $T_p\mathbb{R}^3$ at each $p \in \mathbb{R}^3$. Let $V = \sum f_i E_i$ and calculate $V \cdot E_j = \sum_i f_i \, E_i \cdot E_j = \sum f_i \delta_{ij} = f_j$. thus $V = \sum_i (V \cdot E_i) E_i$. These calculations are done at $p \in \mathbb{R}^3$, but p is arbitrary so we find the result holds for $\mathfrak{X}(\mathbb{R}^3)$ as a module, but of course $V \cdot E_i \in C^\infty(\mathbb{R}^3)$

(2) $V \cdot W = \left(\sum_i f_i E_i\right) \cdot \left(\sum_j g_j E_j\right)$

$$= \sum_{i,j} f_i g_j \underbrace{E_i \cdot E_j}_{\delta_{ij}} = \sum_i f_i g_i \text{ as claimed.}$$

Honestly, I think I already did this somewhere... //

§2.7 Connection Forms

The covariant derivative of a vector field is once more a vector field. At a point $p \in \mathbb{R}^3$ we can express $\nabla_{V_p} W$ as a linear combination of $E_1(P), E_2(P), E_3(P)$ for the frame E_1, E_2, E_3. In particular, we can study the covariant der. of the frame itself in terms of the frame (sort of like the Frenet-Serret Eq$^{\underline{n}}$s)

$$\nabla_{V_p} E_1 = C_{11} E_1(P) + C_{12} E_2(P) + C_{13} E_3(P)$$

$$\nabla_{V_p} E_2 = C_{21} E_1(P) + C_{22} E_2(P) + C_{23} E_3(P)$$

$$\nabla_{V_p} E_3 = C_{31} E_1(P) + C_{23} E_2(P) + C_{33} E_3(P)$$

Applying Lemma 6.3, or common sense of orthonormality,

$$C_{ij} = E_j \cdot (\nabla_{V_p} E_i) = (\nabla_{V_p} E_i) \cdot E_j$$

Recall $\nabla_{aV_p + bW_p} E_i = a \nabla_{V_p} E_i + b \nabla_{W_p} E_i$ and ☆ the dot-product is linear hence C_{ij} is linear in V_p. We define

$$\boxed{\omega_{ij}(V_p) = (\nabla_{V_p} E_i) \cdot (E_j(P)) \quad \text{for } 1 \leq i,j \leq 3}$$

I'm being a bit annoying about V_p verses V in O'neil.

Lemma (7.1) Let E_1, E_2, E_3 be a frame field on $\mathbb{R}^3$. For each $V_p \in T_p\mathbb{R}^3$, let $\omega_{ij}(V_p) = (\nabla_{V_p} E_i) \cdot E_j(P)$ then each ω_{ij} is a one-form and $\omega_{ij} = -\omega_{ji}$. These ω_{ij} are called the connection forms for E_1, E_2, E_3.

Proof: ☆ shows ω_{ij} is one-form for $1 \leq i,j \leq 3$. Consider,

$$0 = V_p[\underbrace{E_i \cdot E_j}_{\delta_{ij}}] = (\nabla_{V_p} E_i) \cdot E_j(P) + E_i(P) \cdot (\nabla_{V_p} E_j) \quad \text{(4) of Th}^{\underline{m}}\ 5.3$$

$$= \omega_{ij}(P) + \omega_{ji}(P). \ //$$

$$\omega_{ij}(V_p) = \underbrace{\left(\nabla_{V_p} E_i\right) \cdot E_j(p)}$$

initial rate of change of E_i in the V_p-direction which goes in the E_j direction. Or, as O'neil says rate at which E_i rotates to E_j as p moves in V_p-direction.

$$\boxed{\begin{array}{l}\text{Th}^{\underline{m}}/(7.2) \text{ Let } \omega_{ij} \text{ be connection forms for frame } E_1, E_2, E_3 \\ \text{on } \mathbb{R}^3. \text{ Then for any vector field } V \text{ on } \mathbb{R}^3, \\ \qquad \underbrace{\nabla_V E_i = \sum_j \omega_{ij}(V) E_j}_{\text{connection eq}^{\underline{n}}\text{'s for } E_1, E_2, E_3 \text{ frame.}}\end{array}}$$

Proof: $\nabla_V E_i = \sum_{j=1}^{3} \left[(\nabla_V E_i) \cdot E_j \right] E_j$ ← $\{E_1, E_2, E_3\}$ form orth. basis.

$= \sum_{j=1}^{3} \omega_{ij}(V) E_j$ ← def$^{\underline{n}}$ of ω_{ij} //

Sorry, I got weary of the "p" see pg. 89 for the "p".

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$\omega_{11} = -\omega_{11}$, $\omega_{22} = -\omega_{22}$, $\omega_{33} = -\omega_{33}$

Thus $\omega_{12}, \omega_{13}, \omega_{23}$ uniquely determine ω

$$\omega = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix}$$

the matrix of connection forms

WAIT, WHAT, A MATRIX OF FORMS. YES. A MATRIX...

Expand the connection eq$^{\text{n}}$'s in terms of the connection matrix coeff. forms $\omega_{12}, \omega_{13}, \omega_{23}$,

$$\left.\begin{aligned} \nabla_V E_1 &= \omega_{12}(V)E_2 + \omega_{13}(V)E_3 \\ \nabla_V E_2 &= -\omega_{12}(V)E_1 + \omega_{23}(V)E_3 \\ \nabla_V E_3 &= -\omega_{13}(V)E_1 - \omega_{23}(V)E_2 \end{aligned}\right\} \quad \underset{\sim}{\text{like}} \quad \begin{aligned} T' &= \kappa N \\ N' &= -\kappa N + \tau B \\ B' &= -\tau N \end{aligned}$$

Oneil notes κ, τ are fncts defined along some curve whereas $\omega_{12}, \omega_{13}, \omega_{23}$ are functions of $V \in \mathfrak{X}(\mathbb{R}^3)$. Moreover, the frame holds $\forall p \in \mathbb{R}^3$, it has no particular application to some an object in $\mathbb{R}^3$, rather, there is much flexibility and we'll be able to tailor a frame to a surface in later chapters 5–6. Or, even to a curve! <u>See Exercise 8</u>

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Now, we work on connecting an arbitrary frame E_1, E_2, E_3 to U_1, U_2, U_3 and find a computationally convenient method to calculate the connection forms for a given frame. The <u>attitude matrix</u> was previously introduced in ~~Chpt 1~~ §2.1.

$$\begin{aligned} E_1 &= a_{11}U_1 + a_{12}U_2 + a_{13}U_3 = \cancel{(a_{11}, a_{12}, a_{13})} \\ E_2 &= a_{21}U_1 + a_{22}U_2 + a_{23}U_3 = \cancel{(a_{21}} \\ E_3 &= a_{31}U_1 + a_{32}U_2 + a_{33}U_3 \end{aligned}$$

Where $a_{ij} = E_i \cdot U_j$ and the <u>attitude matrix</u> A is,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We can prove $A^TA = I$ so $A^T = A^{-1}$ (or $\underset{\text{oneil}}{{}^tA} = A^T$)

Th$^{\underline{m}}$ (7.3) If $A = (a_{ij})$ is the attitude matrix and $\omega = (\omega_{ij})$ is the matrix of connection forms for the frame field E_1, E_2, E_3 then

$$\omega = dA\, A^T \quad \leftarrow \text{matrix multiplication}$$

Explicitly,

$$\omega_{ij} = \sum_k a_{jk}\, da_{ik}$$

Oneil gives an example for cylindrical frame field at this point. I'll attempt a proof, but, first, I must check the notation,

$$\omega_{ij} = (dA\, A^T)_{ij} = \sum_k (dA)_{ik} (A^T)_{kj} = \sum_k A_{jk} (dA)_{ik}$$

But, $(dA)_{ik} = dA_{ik}$ by def$^{\underline{n}}$ and $A_{ik} = a_{ik}$ by def$^{\underline{n}}$ hence we arrive at $\omega_{ij} = \sum_k a_{jk}\, da_{ik}$ as advertised in Th$^{\underline{m}}$.

Proof:
$$\begin{aligned}
\omega_{ij}(v) &= (\nabla_v E_i) \cdot E_j \qquad \text{(suppressing } p\text{-dependence)} \\
&= \nabla_v \left(\sum_{l=1}^{3} a_{il} U_l \right) \cdot E_j \\
&= \sum_{l=1}^{3} \left(v[a_{il}] U_l + a_{il} \nabla_v U_l \right) \cdot \left(\sum_k a_{jk} U_k \right) \\
&= \sum_{l,k} \Big(a_{jk} v[a_{il}] \underbrace{U_l \cdot U_k}_{\delta_{lk}} + a_{il} a_{jk} \cancel{(\nabla_v U_l)} \cdot U_k \Big) \quad \overset{\rightarrow 0}{} \\
&= \sum_k a_{jk}\, da_{ik}(v) \\
&= \left(\sum_k a_{jk}\, da_{ik} \right)(v) \qquad \therefore \ \underline{\omega_{ij} = dA\, A^T} \ //
\end{aligned}$$

constant on $\mathbb{R}^3$ so ∇_v is zero!