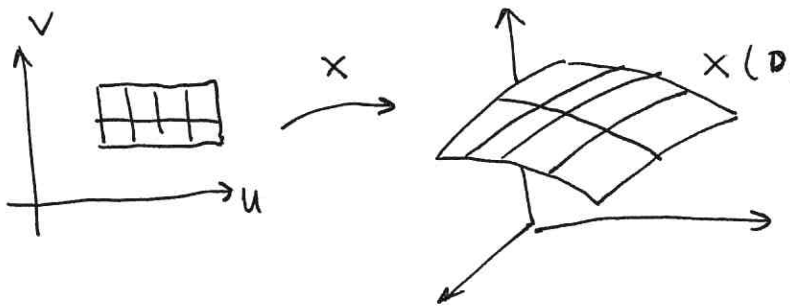
CHAPTER 4 : CALCULUS ON A SURFACE

Def^n/ A coordinate patch $x : D \longrightarrow \mathbb{R}^3$ is a one-one regular mapping of an open set $D \subseteq \mathbb{R}^2$ into $\mathbb{R}^3$

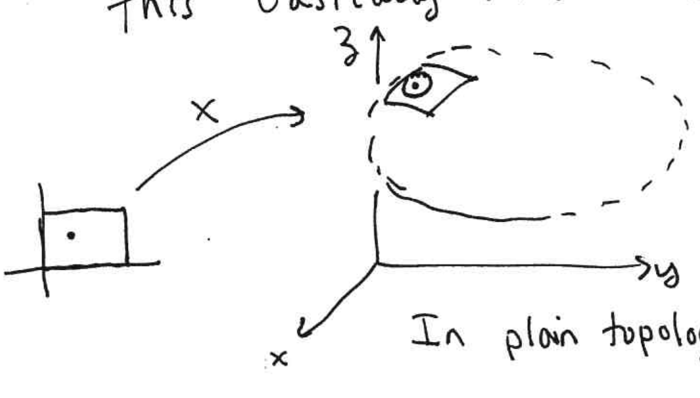
Furthermore, to avoid technicalities we should often like to use "proper patches" for which $x^{-1} : x(D) \longrightarrow D$ is continuous.



<u>Remark</u>: in the study of manifolds it is my custom to use $x : U \subseteq M \longrightarrow \mathbb{R}^n$ as notation for the <u>coordinate chart</u>. That is backwards of Oneil's choice. We're both right. If you primarily use patches then surely you don't want -1 on every patch and vice-versa, since manifold theory (grown-up abstract version) is typically chart-based, the chart notation should be as simple as possible. In my 332 notes (circa Fall 2013) I use $\phi : V \subseteq \mathbb{R}^n \longrightarrow M$ as the patch

Def^n/ A surface in $\mathbb{R}^3$ is a subset M of $\mathbb{R}^3$ such that each point $p \in M$ has a proper patch in M whose image contains a nbhd of p in M.

this basically forbids edges which are not fuzzy.



In plain topological terms $\mathrm{bd}(M) = \emptyset$.

(2)

Example: upper hemisphere of unit-sphere $H(z_+) = \{(x,y,z) \mid z>0,\ x^2+y^2+z^2=1\}$

Let $\Phi(u,v) = (u, v, \sqrt{1-u^2-v^2})$ then

clearly $\Phi : D_2 \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ maps into H, $D_2 = \{(u,v) \mid u^2+v^2<1\}$.

Clearly Φ is 1-1 and the inverse map is simply

$$\Phi^{-1}(x,y,z) = (x,y)$$

which is continuous. It remains to check regularity. (see pg. 39, $F:\mathbb{R}^n \to \mathbb{R}^m$ is regular iff dF_p is 1-1 for all $p \in \mathbb{R}^n$. Likewise regularity on D_2 just means the same but, for all $p \in D_2$ naturally.)

The dF_p is injective iff $\text{rank}(dF_p) = n$ for $F:\mathbb{R}^n \to \mathbb{R}^m$ thus in our $n=2$ context this full-rank condition amounts to checking $\exists$ at least 2 LI vectors amongst the columns of $\Phi' = \left[\frac{\partial \Phi}{\partial u} \middle| \frac{\partial \Phi}{\partial v}\right] \leftarrow$ Jacobian.

$$[d\Phi_p] = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix} \quad \text{for } f = \sqrt{1-u^2-v^2}$$

It is clear $\text{rank}(d\Phi) = 2$ at all $(u,v) \in D_2$.

Moreover, it is clear $d\Phi_p$ exists $\forall p \in D_2$, be careful, his term "mapping" implicts global differentiability of the function. (see pg. 34 under Def^{n} 7.1)

We find $H(z_+)$ is a surface. Moreover, by the natural modifications of this example we can show $H(z_-)$, $H(x_\pm)$, $H(y_\pm)$ are also surfaces. The union of these forms $S_2 \subseteq \mathbb{R}^3$ the unit-sphere.

Advice on checking if $\Sigma : D \to \mathbb{R}^3$ is a proper patch

1.) verify Σ is injective
2.) verify Σ^{-1} is continuous
3.) verify Σ is differentiable, well, smooth in my book. this means $d\Sigma_p$ exists for each $p \in D$.

4.) check regularity by showing $d\Sigma_p$ has rank two for each p. How to do this? Well linear algebra offers us many options:

- LI of two columns
- LI of any two rows

$\Big\}$ $\dim(\text{Row}(\Sigma')) = \dim(\text{Col}\,\Sigma')$.

- at least on $\det(M_p) \neq 0$ where

$$M_p = \underbrace{\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}, \begin{bmatrix} x_u & x_v \\ z_u & z_v \end{bmatrix}, \begin{bmatrix} y_u & y_v \\ z_u & z_v \end{bmatrix}}_{\text{possible } 2\times 2 \text{ natural submatrices of the Jacobian.}}$$

Example: A graph of $f : U \to \mathbb{R}$ is a surface with patch $\Sigma(u,v) = (u, v, f(u,v))$. This Σ is clearly smooth if f is smooth and $\Sigma^{-1}(u,v) = (u,v)$ is continuous and Σ is injective. Thus $z = f(x,y)$ is naturally made a surface by the Monge patch provided $\text{dom}(f)$ is open in $\mathbb{R}^2$.

Remark: a surface like the graph above which is covered by a single patch is called simple. In contrast, the sphere S_2 is not a simple surface. (I'm not so sure proving the assertion S_2 is not simple is so simple though ...)

$Th^{\underline{m}}$/(1.4) Let g be differentiable, $\mathbb{R}$-valued function of $\mathbb{R}^3$ and $c \in \mathbb{R}$ then $g^{-1}\{c\} = \{(x,y,z) \mid g(x,y,z) = c\}$ is a <u>surface</u> provided $\underbrace{dg|_M \neq 0}_{\nexists\, p \in M \text{ for which } dg_p = 0}$.

<u>Proof</u>: If $p \mapsto dg_p$ is continuous then $dg_{p_o}(e_j) \neq 0$ at p_o $\Rightarrow \exists\, U$ open in $\mathbb{R}^3$ for which $dg_p(e_j) \neq 0 \;\; \forall p \in U$. Hence, apply the implicit function theorem to solve for x_j as functions of the remaining x_i $(i \in \mathbb{N}_3 - \{j\})$.

For example, $dg_p(e_3) = \frac{\partial g}{\partial z}(p) \neq 0 \Rightarrow \exists\, h : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ and $g(x,y,h(x,y)) = c \;\; \forall (x,y) \in U$. Thus $X_1(x,y) = (x,y,h_1(x,y))$ gives regular patch near $p_o \in M$. Note, the implicit fnct $Th^{\underline{m}} \Rightarrow$ smoothness of h. Similar arguments give $X_2(x,z) = (x, h_2(x,z), z)$, $X_3(y,z) = (h_3(y,z), y, z)$ Monge-like patches for M. It follows that M is a surface. //

JUST DO $M = g^{-1}\{c\}$ with $dg|_M \neq 0 \Rightarrow M$ surface

<u>Remark</u>: it is possible that $dg|_M \equiv 0$ and yet $M = g^{-1}\{c\}$ is a surface.... is it ??? Can we have $dg_p = 0$ for all $p \in M$ and yet $M = g^{-1}\{c\}$ is a surface? I think not, notice

Suppose M is a surface with $M = g^{-1}\{c\}$ for some smooth function $g : \mathbb{R}^3 \to \mathbb{R}$ and $c \in \mathbb{R}$. Then for each $p \in M$, $\exists X : D \subseteq \mathbb{R}^2 \to M \subseteq \mathbb{R}^3$ which is regular. Note,

$$g(X(u,v)) = c \qquad \forall (u,v) \in D$$

$$\Rightarrow dg \circ dX = 0 \Rightarrow [\partial_x g, \partial_y g, \partial_z g] \begin{bmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \\ \partial_u z & \partial_v z \end{bmatrix} = 0$$

$\vdots$

Example: M_R: $\underbrace{x^2+y^2+z^2}_{g} = \underbrace{R^2}_{c}$

Note $dg = 2x\,dx + 2y\,dy + 2z\,dz \neq 0$ on M_R provided $R > 0$.

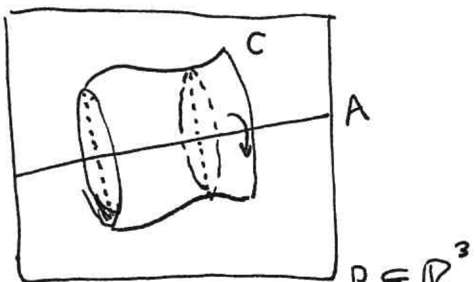
Example: $M : g = c$, $g(x) = \sum_{i=1}^{3}(x_i - c_i)^2$ for constants c_1, c_2, c_3. Then $dg = \sum_{i=1}^{3} 2(x_i - c_i)\,dx_i$

and $dg \neq 0$ unless $x_i = c_i$ which only happens if $c = 0$.

Thus M is surface for $c \neq 0$. This is sphere of radius $\sqrt{c}$ centered at (c_1, c_2, c_3). Note $M = \emptyset$ for $c<0$.

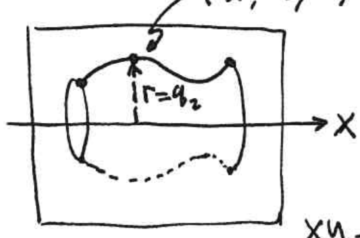
(Oneil calls this Σ_1 on pg. 135)

Example: Surfaces of Revolution



Convenience of exposition, $\ni$ P a coordinate plane and A a coordinate axis, C in $y>0$ half-plane

$(q_1, q_2, 0) \longrightarrow$ rotates to $(q_1, q_2\cos v, q_2 \sin v)$ for $0 \leq v \leq 2\pi$.



Hence, $p \in M$ iff the point $p = (p_1, p_2, p_3)$ and the correponding $\bar{p} = (p_1, \sqrt{p_2^2+p_3^2}, 0)$ is in C

Hence, if $C : f(x,y) = c$ then we can define $g : \mathbb{R}^3 \to \mathbb{R}$ via $g(x,y,z) = f(x, \sqrt{y^2+z^2}) = c$ yields M. Note,

Remark: I did not (yet) find $dg \neq 0$ easy to show.

Continuing to check $dg \neq 0$ for $g = f(x, \sqrt{y^2+z^2})$ (6)

Skip This Page until later

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz$$

$$= \frac{\partial}{\partial x}\left[f(x, \sqrt{y^2+z^2})\right]dx + \frac{\partial}{\partial y}\left[-\right]dy + \frac{\partial}{\partial z}\left[-\right]dz$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\frac{\partial}{\partial y}\sqrt{y^2+z^2} + \frac{\partial f}{\partial y}\frac{\partial}{\partial z}\left[\sqrt{y^2+z^2}\right]$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\left[\frac{y\,dy}{\sqrt{y^2+z^2}} + \frac{z\,dz}{\sqrt{y^2+z^2}}\right]$$

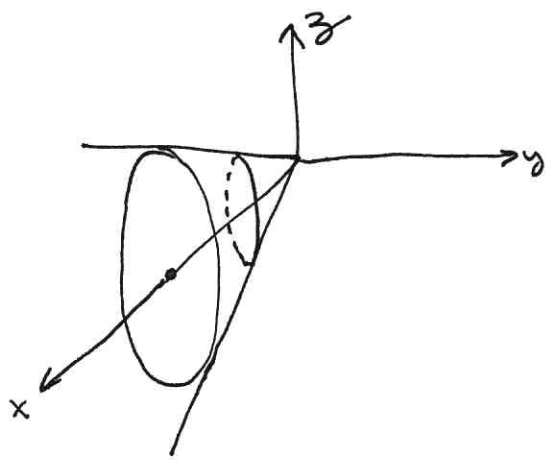
(Now, $f = c$ for $g(x,y,0) = f(x,y)$ with $y > 0$
$\Rightarrow df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$)

If $\left.\frac{\partial f}{\partial x}\right|_p \neq 0$ then $dg_p \neq 0$. Suppose $\left.\frac{\partial f}{\partial x}\right|_p = 0$

then ....

<u>Example</u>: $y = x$ gives line, $f(x,y) = y - x$

$$g(x,y,z) = f(x, \sqrt{y^2+z^2}) = \underbrace{\sqrt{y^2+z^2} - x = 0}_{x = \sqrt{y^2+z^2}}$$



$$dg = \frac{y\,dy + z\,dz}{\sqrt{y^2+z^2}} - dx$$

clearly $\frac{\partial g}{\partial x} = -1 \neq 0$ hence surface ... except at $y = z = 0$ the singularity of the cone.

If $f(x,y) = c$ gives curve then at each point along C we may solve for either x or y hence wlog
$f(x,y) = y - g(x)$ or $f(x,y) = x - h(x)$
hence $dg \neq 0$ as dy or dx have coefficients of 1.

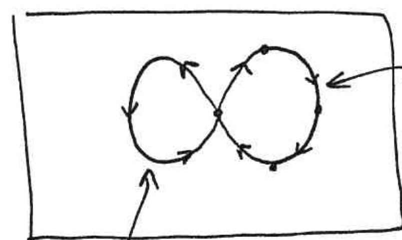
IMPROPER PATCH EXAMPLE

$\mathbf{X}(u,v) = (\sin u, \sin 2u, v)$ is injective. for suitably restricted domain of u.

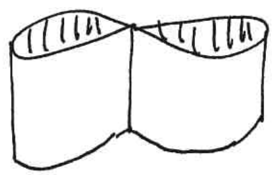
$-\pi < u < \pi$

we can visualize this

	x	y	z
$(0, v)$	0	0	v
$(\pi/2, v)$	1	0	v
$(\pi/4, v)$	$1/\sqrt{2}$	1	v
$\left(\frac{3\pi}{4}, v\right)$	$+1/\sqrt{2}$	-1	v
(π, v)	0	0	v



One for each z,



Since $\mathbf{X}$ is 1-1 it follows $\mathbf{X}^{-1} : \mathbf{X}(\text{dom}(\mathbf{X})) \longrightarrow \text{dom}(\mathbf{X})$ exists. If $\mathbf{X}(u,v) = (a,b) \iff \mathbf{X}^{-1}(a,b) = \mathbf{X}(u,v) = (\underbrace{*, **}_{\text{require}}, v)$

The $*$, $**$ functions will not be continuous.

§4.2 PATCH CALCULATIONS

$u \longmapsto \mathbf{x}(u, v_0)$ gives u-parameter curve

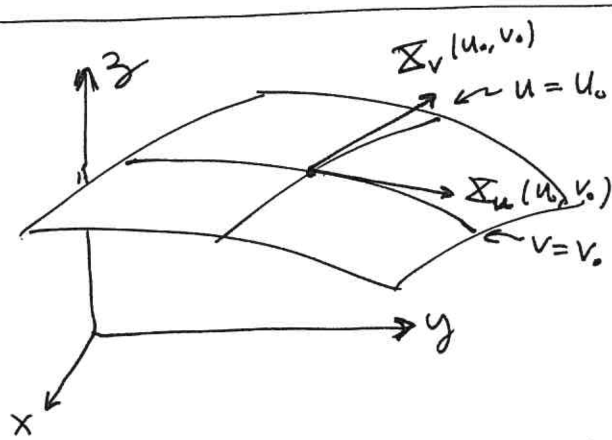
$v \longmapsto \mathbf{x}(u_0, v)$ gives v-parameter curve

Naturally $\mathbf{x}(D)$ is covered by families of u, v-parameter curves. These are images of horizontal & vertical lines in D.

Def^n/ If $\mathbf{x} : D \longrightarrow \mathbb{R}^3$ is a patch, for each $(u_0, v_0) \in D$:

(1.) the velocity vector at u_0 of u-parameter curve, $v = v_0$, is denoted $\mathbf{x}_u(u_0, v_0)$

(2.) likewise for $\mathbf{x}_v(u_0, v_0) = \frac{d}{dv}\left[\mathbf{x}(u_0, v)\right]\Big|_{v = v_0}$.



Note, if $\mathbf{x} = (x, y, z)$ then

$\mathbf{x}_u = (x_u, y_u, z_u)_{\mathbf{x}}$ or $\dfrac{\partial \mathbf{x}}{\partial u} = \dfrac{\partial x}{\partial u}\dfrac{\partial}{\partial x} + \dfrac{\partial y}{\partial u}\dfrac{\partial}{\partial y} + \dfrac{\partial z}{\partial u}\dfrac{\partial}{\partial z}$

$\mathbf{x}_v = (x_v, y_v, z_v)_{\mathbf{x}}$ or $\dfrac{\partial \mathbf{x}}{\partial v} = \dfrac{\partial x}{\partial v}\dfrac{\partial}{\partial x} + \dfrac{\partial y}{\partial v}\dfrac{\partial}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial}{\partial z}$.

(if we ignore Oneil and revert to $U_i(p) = \frac{\partial}{\partial x^i}\Big|_p$ notation)

<u>Remark</u>: See 140 – 145 for more explicit examples; sphere, cones, torus etc...

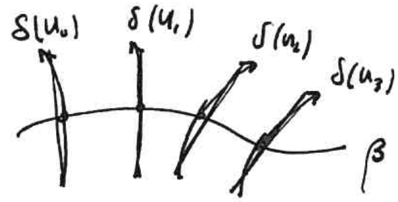
Def^n/ A regular mapping $\mathbf{x} : D \longrightarrow \mathbb{R}^3$ whose image lies in a surface M is called a <u>parametrization</u> of the region $\mathbf{x}(D)$.

(not 1-1 in general, $\mathbf{x}(\theta, z) = (\cos\theta, \sin\theta, z)$ parametrizes the cylinder $x^2 + y^2 = 1$ for $(\theta, z) \in [0, 2\pi] \times \mathbb{R}$.

Remark: Regularity of $X : D \to \mathbb{R}^3$ can be checked by the nontriviality of $\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}$ (you might recall this is normal vector field to surface from calculus III)

Remark: ruled surface is defined on pg. 145.

$$X(u,v) = \underbrace{\beta(u)}_{\text{BASE CURVE}} + v\,\underbrace{\delta(u)}_{\text{DIRECTOR CURVE}}$$

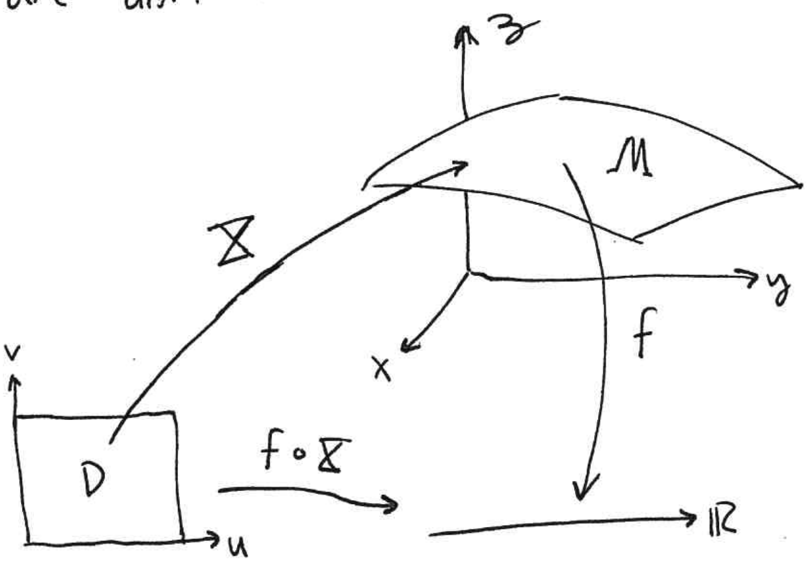


Hmm. PROBLEM 2 on pg. 145 is historic

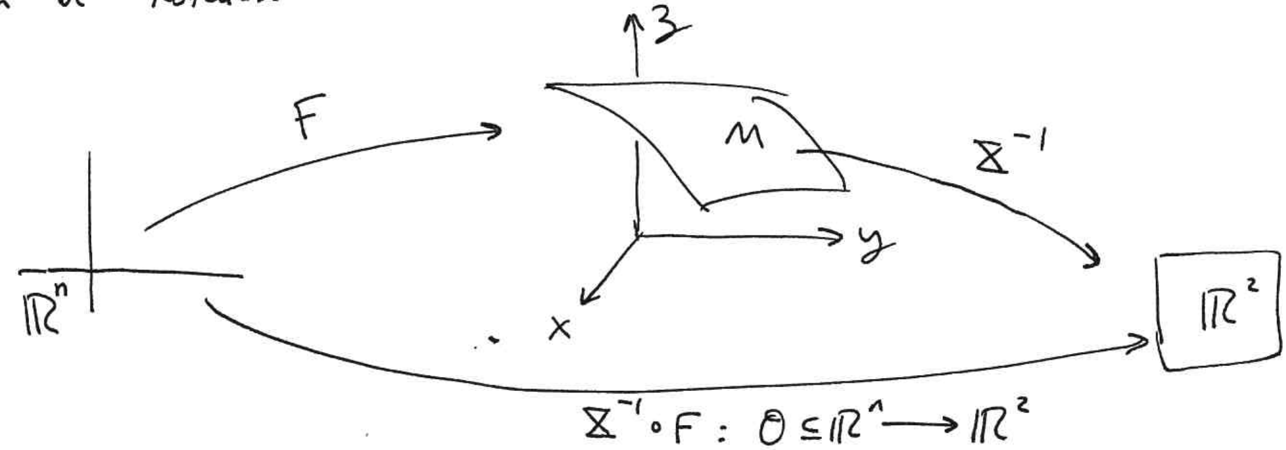
If time/energy permits give a few examples here.

§4.3: DIFFERENTIAL FUNCTIONS AND TANGENT VECTORS

Let $f : M \to \mathbb{R}$ be a function
If $X : D \to M$ is a coordinate patch in M then the composite function $f \circ X$ is called a local coordinate expression for f; $(f \circ X)(u,v) \in \mathbb{R}$ and we could denote this $(u,v) \longmapsto f(X(u,v))$. We say f is differentiable provided all of its coordinate expressions are diff. in the usual Euclidean (advanced calculus) sense

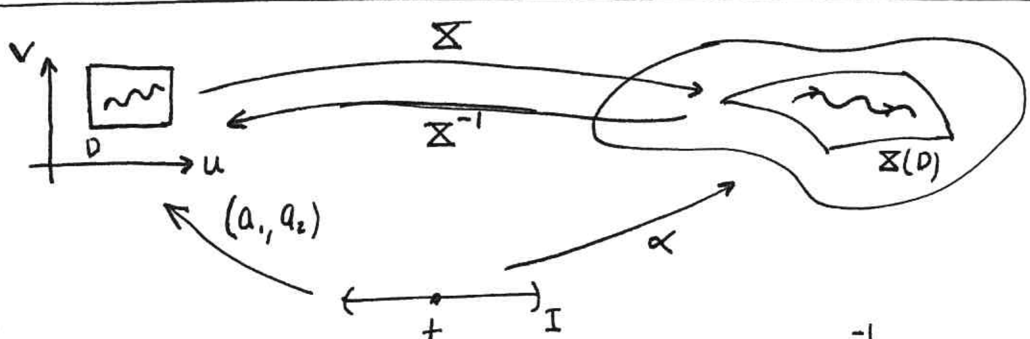


Moving to general maps $F : \mathbb{R}^n \longrightarrow M$. we say F differentiable iff each coord. expression $X^{-1} \circ F$ is diff. on $\mathcal{O}$ where $\mathcal{O}$ is the point-set for which $X^{-1} \circ F$ is well-defined ($F(P) \in X(D)$ so $X^{-1}(F(P))$ is in D). We require $\mathcal{O}$ be open in $\mathbb{R}^n$ so diff. of $X^{-1} \circ F : \mathcal{O} \longrightarrow \mathbb{R}^2$ can be tested...



Special, important case, <u>CURVE</u> $\alpha : I \longrightarrow M$ is a diff. function from an open-interval I to M.

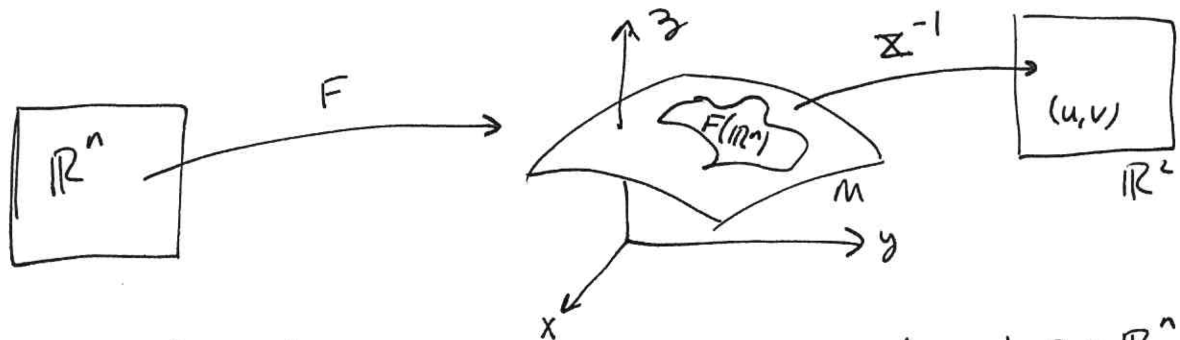
<u>Lemma (3.1)</u> If α is curve $\alpha : I \longrightarrow M$ whose image $\alpha(I) \subset X(D)$ for a single coord. patch X then $\exists!$ diff functions $a_1, a_2 : I \longrightarrow \mathbb{R}$ such that $\alpha(t) = X(a_1(t), a_2(t))$. That is; $\alpha = X(a_1, a_2)$ in function notation.



Since $\alpha(I) \subset X(D)$ we may write $\alpha = X X^{-1} \alpha = X(a_1, a_2)$ that is, define $X^{-1} \circ \alpha = (a_1, a_2) \Leftrightarrow X(a_1, a_2) = \alpha$. If $\alpha = X(b_1, b_2)$ then injectivity of X immediately yields $(a_1, a_2) = (b_1, b_2)$. //

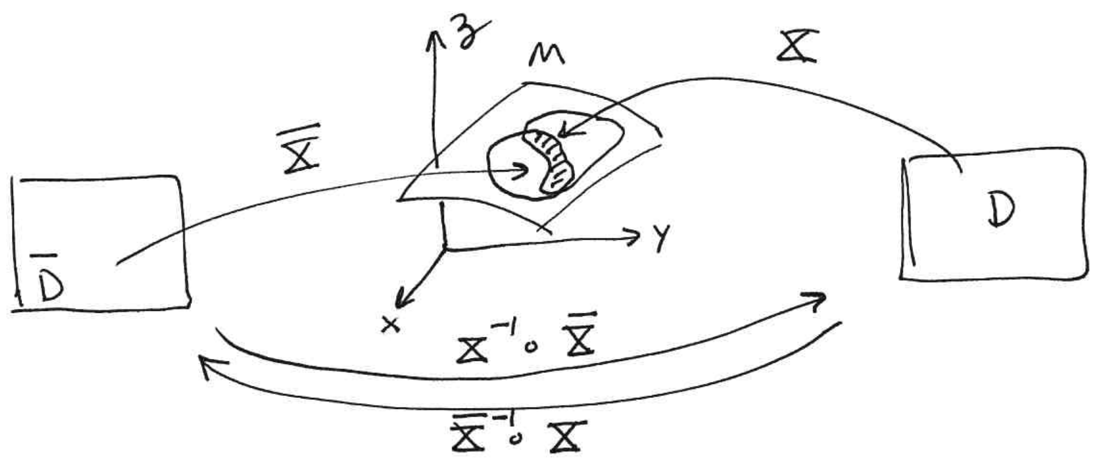
Th$^{\underline{m}}$ (3.2) Let M be a surface in $\mathbb{R}^3$. If $F: \mathbb{R}^n \longrightarrow \mathbb{R}^3$ is a differentiable mapping whose image lies in M, then considered as a function $F: \mathbb{R}^n \longrightarrow M$ (into M), F is diff. function to M.

Proof:



$F: \mathbb{R}^n \longrightarrow \mathbb{R}^3$ differentiable $\Rightarrow$ dF_p exists at each $p \in \mathbb{R}^n$.
$F(\mathbb{R}^n) \subset M$. We need to show $\mathbb{X}^{-1} \circ F: \mathcal{O} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^2$ is differentiable. Also, if $\mathbb{X}(D)$ does not cover $F(\mathbb{R}^n)$ then we'll need to use diff coord. patch as we examine different points... these are the technical details we'd need to sort through... //

Comment: A curve in $\mathbb{R}^3$ which has $\alpha(I) \subset M$ is a curve in M.



Th$^{\underline{m}}$ (3.2) $\Rightarrow$ smooth overlap of patches.

Cor (3.4) If Σ, Υ are overlapping patches on M then $\exists!$ diff fncts $\bar{u}, \bar{v}$ such that ~~Σ to~~

$$\Upsilon(u,v) = \Sigma(\bar{u}(u,v), \bar{v}(u,v)).$$

$\forall (u,v) \in \mathrm{dom}(\Sigma^{-1} \circ \Upsilon)$.

Thus, it suffices to check diff at $p \in M$ for just one coord. patch which covers $p \in M$.

Def^n/ Let p be a point on surface M in $\mathbb{R}^3$. A tangent vector v to $\mathbb{R}^3$ at p is <u>tangent to M</u> at p provided $\exists \alpha : I \to M$ for which $\alpha'(0) = v_p$. All these form T_pM.

<u>Lemma (3.6)</u>: Consider $p \in M$ for which $p \in \Sigma(D)$ and $\Sigma(u_0, v_0) = p$. Then $v_p \in T_pM$ iff $v_p \in \mathrm{span}\{\Sigma_u(p), \Sigma_v(p)\}$.

<u>Proof</u>: in lecture. or pg. 153.//

Def^n (3.7) A Euclidean vector field Z on M in $\mathbb{R}^3$ is a function which assigns for each $p \in M$ a vector $Z(p) \in T_p\mathbb{R}^3$. If $Z(p) \in T_pM$ $\forall p \in M$ then Z is <u>tangent</u> to M. $Z(p) \in (T_pM)^\perp$ $\forall p \in M$ is said to be <u>normal</u> to M.

Lemma (3.6) Let $p \in M$ a surface in $\mathbb{R}^3$ and X a patch in M s.t. $X(u_o, v_o) = p$. Then $V_p \in T_pM$ iff V_p is a linear combination of $X_u(u_o, v_o)$, $X_v(u_o, v_o)$

Proof: Observe $X_u, X_v \in T_pM$ as $X_u = \alpha'(u_o)$ and $X_v = \beta'(v_o)$ where $\alpha(u) = X(u, v_o)$ and $\beta = X(u_o, v)$.

We defined T_pM via tangents to curves. Let $V_p \in T_pM$ then by def$^{\underline{n}}$ $\exists \alpha : I \to M$ with $\alpha'(0) = V_p$. Moreover, by Lemma 3.1, $\alpha = X(a_1, a_2)$

$$\begin{aligned}
\alpha' &= \frac{d}{dt}\left[X(a_1, a_2)\right] \\
&= \frac{\partial X}{\partial u}(a_1, a_2)\frac{da_1}{dt} + \frac{\partial X}{\partial v}(a_1, a_2)\frac{da_2}{dt}
\end{aligned}$$

$$\Rightarrow \alpha'(0) = a_1'(0)\, X_u(u_o, v_o) + a_2'(0)\, X_v(u_o, v_o)$$

Thus $V_p \in \text{span}\{X_u, X_v\}$ at (u_o, v_o). Conversely if $\exists c_1, c_2$ s.t. $V = c_1 X_u(u_o, v_o) + c_2 X_v(u_o, v_o)$ observe that $\alpha(t) = X(u_o + tc_1, v_o + tc_2)$ has $\alpha'(0) = V$ hence $\text{span}\{X_u(u_o, v_o), X_v(u_o, v_o)\} \subseteq T_pM$ & $T_pM \subseteq \text{span}\{\ \}$

We conclude $\text{span}\{X_u(u_o, v_o), X_v(u_o, v_o)\} = T_pM$.

Remark: the differential notation for tangent vectors gives a nice notational guide to T_pM, $X = (x, y, z)$ then,

$$\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial}{\partial z}$$

Lemma (3.8) If $M : g = c$ is a surface in $\mathbb{R}^3$, then the gradient vector field $\nabla g = \sum \frac{\partial g}{\partial x_i} U_i$ restricted to M is a nonvanishing normal vector field on M.

Proof: If $\alpha : I \longrightarrow M$ is curve on M and we consider $g(\alpha(t)) = c$ $\forall t \in I$ then diff. and use chain rule to find $\nabla g \cdot \alpha' = 0$ or more precisely

$$\nabla g\,(\alpha(t)) \cdot \alpha'(t) = 0$$

Hence, for $p = \alpha(0)$, $\nabla g(p) \cdot V_p = 0$ for $V_p \in T_pM$.
But, α is arbitrary $\Rightarrow$ $\nabla g(p) \cdot V_p = 0$ $\forall V_p \in T_pM$ hence $\nabla g(p) \in T_pM^{\perp}$.

Remark: I mentioned $\nexists$ a nonvanishing vectorfield on the sphere Σ, but, we see now the normal vector field to Σ is non vanishing,

$$g = x^2+y^2+z^2 \Rightarrow \underbrace{\nabla g = \langle 2x, 2y, 2z\rangle}_{\text{not tangent to the sphere so no } \rightarrow\leftarrow \text{ to my claim.}} \neq 0$$

Def^{n} (3.10) Let V be a tangent vector to M at p and let f be diff. $\mathbb{R}$-valued fnct. on M. Then $V[f]$ is the derivative of f w.r.t. V is given by the common value $\frac{d}{dt}\left[f \circ \alpha\right]\Big|_{t=0}$ for all curves α in M with $\alpha'(0) = V$.

Let's see $V = c_1 \mathbf{x}_u + c_2 \mathbf{x}_v$ then $V[f] = c_1 \frac{\partial f}{\partial u} + c_2 \frac{\partial f}{\partial v}$ I hope.
Curious, not much discussion here... perhaps he returns to it later...

Two ways to describe curve on M:

- IMPLICITLY: $g(\alpha(t)) = c \longrightarrow \nabla g(\alpha(t)) \cdot \alpha'(t) = 0$
- EXPLICITLY: $\alpha = \mathbf{X}(a_1, a_2) \longrightarrow \alpha' = \mathbf{X}_u a_1' + \mathbf{X}_v a_2'$

§4.4: DIFFERENTIAL FORMS ON SURFACE

0-form on M is function from M to $\mathbb{R}$.
1-form on M is $\phi_p : T_pM \longrightarrow \mathbb{R}$, $p \longmapsto \phi_p$ for each $p \in M$.
2-form on M is $\eta : T_pM \times T_pM \longrightarrow \mathbb{R}$ with η bilinear and,

$$\eta(v,w) = -\eta(w,v).$$

Notice if $V, W \in \mathfrak{X}(M)$ then $\phi(V)$, $\eta(V,W)$ are functions on M. For future convenience, let's denote $\Lambda^p M$ as p-forms on M and $\Omega(M) = \Lambda^0(M) \oplus \Lambda^1(M) \oplus \Lambda^2(M) \Leftarrow$ exterior algebra on M which forms a $C^\infty(M)$-module. Wedge products and determinants are nearly interchangeable. For example:

Lemma (4.2) Let η be a 2-form on M and let $v_p, w_p \in T_pM$ then $\eta_p(av_p + bw_p, cv_p + dw_p) = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \eta_p(v_p, w_p)$

Proof: drop the p-dependence

$$\begin{aligned}\eta(av+bw, cv+dw) &= a\,\eta(v, cv+dw) + b\,\eta(w, cv+dw) \\ &= ac\,\eta(v,v) + ad\,\eta(v,w) + bc\,\eta(w,v) + bd\,\eta(w,w) \\ &= (ad-bc)\,\eta(v,w). //\end{aligned}$$

For fun, in $n=2$ case the $\sum$ loses.

$$\begin{aligned}\eta\left(\sum x^i e_i, \sum y^j e_j\right) &= \sum_i \sum_j x^i y^j \, \eta(e_i, e_j) \\ &= \sum_i \sum_j \epsilon_{ij} x^i y^j \, \eta(e_1, e_2) \\ &= \det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \eta(e_1, e_2).\end{aligned}$$

Wedge Product

for 0-forms we simply multiply. However, for one-forms,

Def$^{\underline{n}}$/ If $\phi, \psi \in \Lambda^1(M)$ then $\phi \wedge \psi \in \Lambda^2(M)$ defined by

$$(\phi \wedge \psi)(v, w) = \phi(v)\psi(w) - \psi(v)\phi(w)$$

For all v, w tangent to M.

Note it is clear $\phi \wedge \psi$ is bilinear and skew-symmetric given the def$^{\underline{n}}$ above hence $\phi \wedge \psi \in \Lambda^2(M)$ is reasonable.

$$\xi \wedge \eta = (-1)^{pq} \eta \wedge \xi$$

So $p = 0, 2$ forms commute with $p = 0, 1, 2$ forms and only $p = 1, q = 1$ gives $\alpha \wedge \beta = -\beta \wedge \alpha$.

Exterior Derivative:

0-form, $f : M \to \mathbb{R}$ $\xrightarrow{d}$ $df \in \Lambda^1(M)$ where $df_p(V_p) = V_p[f]$ as before, we require no new special def$^{\underline{n}}$ (of course, $V_p \in T_pM$ so this is not exactly the same as the $\mathbb{R}^2, \mathbb{R}^3$ - exterior calculus of Chpt 1)

1-form, $\phi \in \Lambda^1(M)$ $\xrightarrow{d}$ $d\phi \in \Lambda^2(M)$.

Def$^{\underline{n}}$/ $d\phi(X_u, X_v) = \frac{\partial}{\partial u}(\phi(X_v)) - \frac{\partial}{\partial v}(\phi(X_u))$ then extend linearly. If $X(D) = M$ then that is all, otherwise do this for each patch until $d\phi$ is defined on M

Clearly there is a gap to fill here, why does this definition work for points where $\exists$ multiple patches?

$\boxed{\text{Th}^{m} \text{ (4.6): If } f : M \to \mathbb{R} \text{ is a function then } d(df) = 0.}$

Proof: $d(df)(\Phi_u, \Phi_v) = \frac{\partial}{\partial u}\left[df(\Phi_v)\right] - \frac{\partial}{\partial v}\left[df(\Phi_u)\right]$

$$= \frac{\partial}{\partial u}\left[\frac{\partial f}{\partial v}\right] - \frac{\partial}{\partial v}\left[\frac{\partial f}{\partial u}\right]$$

$$= 0.$$

But, as Φ was an arbitrary patch on $M \Rightarrow d(df) = 0.$ //

Matching Lemmas:

(1.) $\underbrace{\phi = \psi}_{\text{1-forms on } X(D)} \iff \left\{ \begin{array}{l} \phi(\Phi_u) = \psi(\Phi_u) \\ \phi(\Phi_v) = \psi(\Phi_v) \end{array} \right\}$

(2.) $\underbrace{\mu = \nu}_{\text{2-forms on } X(D)} \iff \mu(\Phi_u, \Phi_v) = \nu(\Phi_u, \Phi_v)$

Differential Calculus on $\mathbb{R}^2$ as surface

f a fnct, ϕ a 1-form, η a 2-form then

(1.) $\phi = f_1\, du_1 + f_2\, du_2$, $f_i = \phi(U_i)$.

(2.) $\eta = g\, du_1 \wedge du_2$, $g = \eta(U_1, U_2)$.

(3.) For $\psi = g_1\, du_1 + g_2\, du_2$ and ϕ as above,

$$\phi \wedge \psi = (f_1 g_2 - f_2 g_1)\, du_1 \wedge du_2$$

(4.) $df = \frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2$

(5.) $d\phi = \left(\frac{\partial f_2}{\partial u_1} - \frac{\partial f_1}{\partial u_2}\right) du_1 \wedge du_2$

Def^{n}/ A diff form ϕ is closed iff $d\phi = 0$.
A diff form ψ is exact iff $\exists \mu$ s.t $d\mu = \psi$.

Only one-forms are interesting for 2-dim'l case... does exact $\Rightarrow$ closed? YES, $d^2 = 0$. The converse leads to deep topological discussion...

(THERE ARE MISTAKES ?)

$\boxed{\text{Lemma (4.5)}}$ $d\phi(X_u, X_v) = \frac{\partial}{\partial u}\left(\phi(X_v)\right) - \frac{\partial}{\partial v}\left(\phi(X_u)\right)$
is coordinate independent.

CAUTION, MISTAKES BELOW...

Proof: Suppose $\bar{X}(\bar u, \bar v)$ is another patch then chain rule gives,

$$\bar{X}_{\bar u} = \frac{\partial u}{\partial \bar u} X_u + \frac{\partial v}{\partial \bar u} X_v \qquad (1.)$$

$$\bar{X}_{\bar v} = \frac{\partial u}{\partial \bar v} X_u + \frac{\partial v}{\partial \bar v} X_v \qquad (2.)$$

But then as $d\phi$ was clearly a two-form we have

$$d\phi(\bar X_{\bar u}, \bar X_{\bar v}) = d\phi\left(\frac{\partial u}{\partial \bar u} X_u + \frac{\partial v}{\partial \bar u} X_v,\ \frac{\partial u}{\partial \bar v} X_u + \frac{\partial v}{\partial \bar v} X_v\right)$$

$$= \underbrace{\det\begin{pmatrix} \partial u/\partial \bar u & \partial u/\partial \bar v \\ \partial v/\partial \bar u & \partial v/\partial \bar v \end{pmatrix}}_{J} d\phi(X_u, X_v)$$

Therefore, we need the equation above to hold when we apply def^k of $d\phi$. This means it suffices that

$$\frac{\partial}{\partial \bar u}\left[\phi(\bar X_{\bar v})\right] - \frac{\partial}{\partial \bar v}\left[\phi(\bar X_{\bar u})\right] = J\left(\frac{\partial}{\partial u}\left[\phi(X_v)\right] - \frac{\partial}{\partial v}\left[\phi(X_u)\right]\right) \qquad **$$

If we can prove $**$ then consistency of $d\phi$ w.r.t patch-ambiguity is shown... using (1) & (2),

$$\phi(\bar X_{\bar u}) = \frac{\partial u}{\partial \bar u}\phi(X_u) + \frac{\partial v}{\partial \bar u}\phi(X_v) \qquad (\tilde{1}.)$$

$$\phi(\bar X_{\bar v}) = \frac{\partial u}{\partial \bar v}\phi(X_u) + \frac{\partial v}{\partial \bar v}\phi(X_v) \qquad (\tilde{2}.)$$

Then diff. with respect to $\bar u$ & $\bar v$, use product rule

$$\frac{\partial}{\partial \bar u}\left[\phi(\bar X_{\bar v})\right] = \cancel{\frac{\partial^2 u}{\partial \bar u \partial \bar v}\phi(X_u)} + \frac{\partial u}{\partial \bar u}\frac{\partial}{\partial \bar u}\left[\phi(X_u)\right] + \cancel{\frac{\partial^2 v}{\partial \bar v \partial \bar u}\phi(X_v)} + \frac{\partial v}{\partial \bar v}\frac{\partial}{\partial \bar u}\phi(X_v)$$

$$-\ \frac{\partial}{\partial \bar v}\left[\phi(\bar X_{\bar u})\right] = \cancel{\frac{\partial^2 u}{\partial \bar u \partial \bar v}\phi(X_u)} + \frac{\partial u}{\partial \bar u}\frac{\partial}{\partial \bar v}\left[\phi(X_u)\right] + \cancel{\frac{\partial^2 v}{\partial \bar v \partial \bar u}\phi(X_v)} + \frac{\partial v}{\partial \bar v}\frac{\partial}{\partial \bar v}\left[\phi(X_v)\right]$$

$$\frac{\partial}{\partial \bar u}\left[\phi(\bar X_{\bar v})\right] - \frac{\partial}{\partial \bar v}\left[\phi(\bar X_{\bar u})\right] = \frac{\partial u}{\partial \bar u}\frac{\partial}{\partial \bar u}\left[\phi(X_u)\right] - \frac{\partial u}{\partial \bar u}\frac{\partial}{\partial \bar v}\left[\phi(X_u)\right] + \frac{\partial v}{\partial \bar v}\left(\frac{\partial}{\partial \bar u}\left[\phi(X_v)\right] - \frac{\partial}{\partial \bar v}\left[\phi(X_v)\right]\right)$$

$$= \frac{\partial u}{\partial \bar u}\left(\frac{\partial}{\partial \bar u}\left[\phi(X_u)\right] - \frac{\partial}{\partial \bar v}\left[\phi(X_u)\right]\right) + \frac{\partial v}{\partial \bar v}\left(\frac{\partial}{\partial \bar u}\phi(X_v) - \frac{\partial}{\partial \bar v}\phi(X_v)\right)$$

Continuing, (on **) see RHS.

$$RHS = \left(\frac{\partial u}{\partial \bar{u}}\frac{\partial v}{\partial \bar{v}} - \frac{\partial u}{\partial \bar{v}}\frac{\partial v}{\partial \bar{u}}\right)\left(\frac{\partial}{\partial u}\left[\phi(X_v)\right] - \frac{\partial}{\partial v}\left[\phi(X_u)\right]\right)$$

$$= \underbrace{\frac{\partial u}{\partial \bar{u}}\frac{\partial v}{\partial \bar{v}}\frac{\partial}{\partial u}\left[\phi(X_v)\right]}_{\mathrm{I}} - \underbrace{\frac{\partial u}{\partial \bar{u}}\frac{\partial v}{\partial \bar{v}}}_{\mathrm{II}}\frac{\partial}{\partial v}\left[\phi(X_u)\right]$$

$$- \underbrace{\frac{\partial u}{\partial \bar{v}}\frac{\partial v}{\partial \bar{u}}}_{\mathrm{III}}\frac{\partial}{\partial u}\left[\phi[X_v]\right] + \frac{\partial u}{\partial \bar{v}}\frac{\partial v}{\partial \bar{u}}\frac{\partial}{\partial v}\left[\phi(X_u)\right]$$

$$LHS = \frac{\partial u}{\partial \bar{u}}\left(\frac{\partial}{\partial \bar{u}} - \frac{\partial}{\partial \bar{v}}\right)\left[\phi(X_u)\right] + \frac{\partial v}{\partial \bar{v}}\left(\frac{\partial}{\partial \bar{u}} - \frac{\partial}{\partial \bar{v}}\right)\left[\phi(X_v)\right]$$

$$= \frac{\partial u}{\partial \bar{u}}\left(\frac{\partial u}{\partial \bar{u}}\frac{\partial}{\partial u} + \frac{\partial v}{\partial \bar{u}}\frac{\partial}{\partial v} - \frac{\partial u}{\partial \bar{v}}\frac{\partial}{\partial u} - \frac{\partial v}{\partial \bar{v}}\frac{\partial}{\partial v}\right)\left[\phi(X_u)\right]$$

$$+ \frac{\partial v}{\partial \bar{v}}\left(\frac{\partial u}{\partial \bar{u}}\frac{\partial}{\partial u} + \frac{\partial v}{\partial \bar{u}}\frac{\partial}{\partial v} - \frac{\partial u}{\partial \bar{v}}\frac{\partial}{\partial u} - \frac{\partial v}{\partial \bar{v}}\frac{\partial}{\partial v}\right)\left[\phi(X_v)\right]$$

should match.

$$=$$

A notation I should probably use more, the "hook)

$$\phi(X) = X \lrcorner \phi$$

Then we could define, (if we used V, W as derivations)

$$d\phi(V, W) = V(W \lrcorner \phi) - W(V \lrcorner \phi).$$

Coordinate-dependence of $d\phi$

Let's prove there isn't any! Suppose X, Y are patches and $Y(u,v) = X(\bar{u}(u,v), \bar{v}(u,v))$ then note

$$\frac{\partial Y}{\partial u} = \frac{\partial X}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial u} + \frac{\partial X}{\partial \bar{v}}\frac{\partial \bar{v}}{\partial u}$$

$$\frac{\partial Y}{\partial v} = \frac{\partial X}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial v} + \frac{\partial X}{\partial \bar{v}}\frac{\partial \bar{v}}{\partial v}$$

$$d\phi(Y_u, Y_v) = \frac{\partial}{\partial u}\left(\phi(Y_v)\right) - \frac{\partial}{\partial v}\left(\phi(Y_u)\right)$$

$$= \frac{\partial}{\partial u}\left(\phi\left(\frac{\partial X}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial v} + \frac{\partial X}{\partial \bar{v}}\frac{\partial \bar{v}}{\partial v}\right)\right) - \frac{\partial}{\partial v}\left(\phi\left(\frac{\partial X}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial u} + \frac{\partial X}{\partial \bar{v}}\frac{\partial \bar{v}}{\partial u}\right)\right)$$

$$= \cancel{\frac{\partial \bar{u}}{\partial v}}$$

$$W = W[u]\frac{\partial}{\partial u} + W[v]\frac{\partial}{\partial v} = W[\bar{u}]\frac{\partial}{\partial \bar{u}} + W[\bar{v}]\frac{\partial}{\partial \bar{v}}$$

$$= \left(W[\bar{u}]\frac{\partial u}{\partial \bar{u}} + W(\bar{v})\frac{\partial u}{\partial \bar{v}}\right)\frac{\partial}{\partial u}$$

$$+ \left(W(\bar{u})\frac{\partial v}{\partial \bar{u}} + W(\bar{v})\frac{\partial v}{\partial \bar{v}}\right)\frac{\partial}{\partial v}$$

$$\frac{\partial}{\partial \bar{u}} = \frac{\partial u}{\partial \bar{u}}\frac{\partial}{\partial u} + \frac{\partial v}{\partial \bar{u}}\frac{\partial}{\partial v}$$

$$d\phi\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = \frac{\partial}{\partial u}\left(\phi\left(\frac{\partial}{\partial v}\right)\right) - \frac{\partial}{\partial v}\left(\phi\left(\frac{\partial}{\partial u}\right)\right)$$

$$d\phi\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = d\phi\left(\frac{\partial \bar{u}}{\partial u}\frac{\partial}{\partial \bar{u}} + \frac{\partial \bar{v}}{\partial u}\frac{\partial}{\partial \bar{v}}, \frac{\partial \bar{u}}{\partial v}\frac{\partial}{\partial \bar{u}} + \frac{\partial \bar{v}}{\partial v}\frac{\partial}{\partial \bar{v}}\right) = \det\begin{bmatrix} \partial\bar{u}/\partial u & \partial\bar{u}/\partial v \\ \partial\bar{v}/\partial u & \partial\bar{v}/\partial v \end{bmatrix} d\phi\left(\frac{\partial}{\partial \bar{u}}, \frac{\partial}{\partial \bar{v}}\right)$$

$$= \left(\frac{\partial \bar{u}}{\partial u}\frac{\partial \bar{v}}{\partial v} - \frac{\partial \bar{u}}{\partial v}\frac{\partial \bar{v}}{\partial u}\right) d\phi\left(\frac{\partial}{\partial \bar{u}}, \frac{\partial}{\partial \bar{v}}\right)$$

$$d\phi(\partial_u, \partial_v) = \partial_u[\phi(\partial_v)] - \partial_v[\phi(\partial_u)]$$

$$d\phi(\bar\partial_{\bar u}, \bar\partial_{\bar v}) = \bar\partial_{\bar u}[\phi(\bar\partial_{\bar v})] - \bar\partial_{\bar v}[\phi(\bar\partial_{\bar u})]$$

$$d\phi(\partial_u, \partial_v) = \left(\frac{\partial \bar u}{\partial u}\frac{\partial \bar v}{\partial v} - \frac{\partial \bar u}{\partial v}\frac{\partial \bar v}{\partial u}\right) d\phi\left(\frac{\partial}{\partial \bar u}, \frac{\partial}{\partial \bar v}\right)$$

$$= \left(\frac{\partial \bar u}{\partial u}\frac{\partial \bar v}{\partial v} - \frac{\partial \bar u}{\partial v}\frac{\partial \bar v}{\partial u}\right)\left(\frac{\partial}{\partial \bar u}\left[\phi\left(\frac{\partial}{\partial \bar v}\right)\right] - \frac{\partial}{\partial \bar v}\left[\phi\left(\frac{\partial}{\partial \bar u}\right)\right]\right)$$

$$= \frac{\partial \bar u}{\partial u}\,\phi\left(\frac{\partial \bar v}{\partial v}\frac{\partial}{\partial \bar v}\right)$$