CHAPTER 5: SHAPE OPERATORS:

Suppose $Z \in \mathfrak{X}(M)$ then $\nabla_V Z$ is sensible provided $V \in \mathfrak{X}(M)$. As usual $\nabla_V Z$ is the rate of change of Z in the V direction. Two methods to calculate it:

METHOD 1: (CURVES)

Let $\alpha : I \to M$ with $\alpha'(0) = V$.
Let Z_α be restriction of Z to α; $t \mapsto Z(\alpha(t))$
then,

$$\nabla_V Z = (Z_\alpha)'(0)$$

METHOD 2: (COORDINATES, FRAMES)

Express $Z = \sum z_i U_i$ then $\nabla_V Z = \sum_{i=1}^{3} V[z_i]\, U_i$

<u>Are these equivalent?</u> Let's calculate,

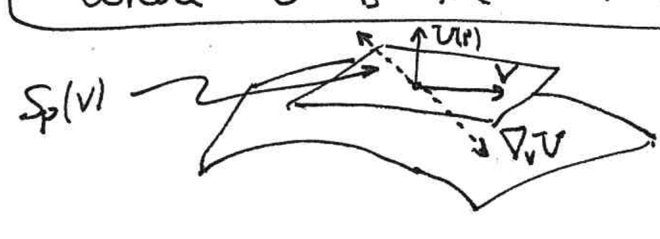
$$(Z_\alpha)'(0) = \sum (z_i(\alpha))'(0)\, U_i = \sum_{i=1}^{3} V[z_i]\, U_i .$$

<u>Comment</u>: $(\nabla_V Z)_p \notin T_pM$ even when $V(p) \in T_pM$. The change might point off of T_pM. However, $\exists$ just ~~two~~ one way to go: $U(p)$. We know

$$T_p\mathbb{R}^3 = T_pM \oplus (T_pM)^\perp \quad \text{and} \quad T_pM^\perp = \text{span}\{U(p)\}$$

where $U(p)$ is <u>normal</u> to T_pM.

> **Def^n/ <u>SHAPE OPERATOR</u>**: For $p \in M$, $V_p \in T_pM$ let
> $$S_p(V) = -\nabla_V U$$
> where U is the unit-normal vector field to M at p.



$S_p(V)$ measures how U turns away from itself in V-direction

Lemma 1.2: For each $p \in M \subset \mathbb{R}^3$, the shape operator is a linear operator; $S_p : T_pM \longrightarrow T_pM$ on T_pM

Proof: $U \cdot U = 1$ on M thus,

$$0 = v_p[U \cdot U] = 2(\nabla_v U)\big|_p \cdot U\big|_p = -2\, S_p(v) \cdot U(p)$$

thus $U_p \cdot S_p(v) = 0 \quad \forall v \in T_pM \Rightarrow S_p(v) \in (\text{span}\, U(p))^{\perp}$

Which means that $S_p(v) \in T_pM$.

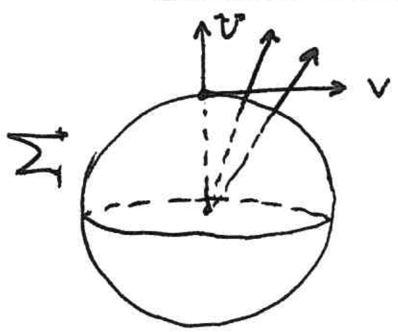
$(V = W_1 \oplus W_2$ then $W_2^{\perp} = W_1$ and $W_1^{\perp} = W_2)$

Examples:

(1) Σ of radius R has $U(p) = \frac{1}{R}\sum_{i=1}^{3} x_i U_i$

thus $\nabla_v U = \frac{1}{R}\sum_{i=1}^{3} v[x_i]\, U_i = \sum_{i=1}^{3} \frac{1}{R} v_i U_i = \frac{v}{R}$

Hence $\boxed{S_p(v) = \frac{-v}{R}}$ $\forall p \in \Sigma = \underbrace{S_2(R)}_{x^2+y^2+z^2=R^2}$.



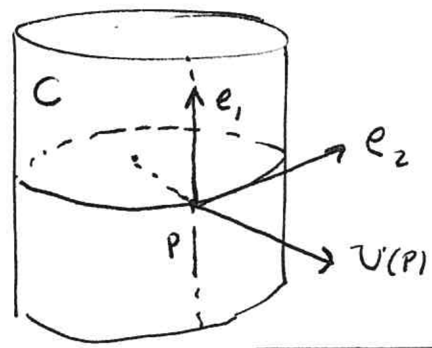
$$U + \Delta U = U_{new} = U + \frac{v}{R}$$

(2.) $\mathcal{P}$ a plane in $\mathbb{R}^3$ has fixed $U(p) = U_0 \quad \forall p \in \mathcal{P}$

$\therefore \quad S_p(v) = -\nabla_v U = 0.$

The tangent planes do not bend on $\mathcal{P}$.

(3.) $C: \; x^2 + y^2 = R^2$, z free



$$\nabla_{e_1} U = 0$$

$$\nabla_{e_2} U = \frac{e_2}{R}$$

$$e_1 = (0, 0, 1)$$

$$e_2 = (\cos\theta, \sin\theta, 0) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0\right)$$

$$e_1 \times e_2 = (-\sin\theta, \cos\theta, 0) = U$$

$$\nabla_{e_2} U = e_2[-\sin\theta]\, U_1 + e_2[\cos\theta]\, U_2$$

(4.) $M: \; z = xy$ consider $P = (0,0,0)$.

$$X(u,v) = (u, v, uv)$$

$$X(x,y) = (x, y, xy)$$

$$X(s,t) = (s, t, st) \leftarrow \text{I'll go with this.}$$

$$\frac{\partial X}{\partial s} = (1, 0, t), \quad \frac{\partial X}{\partial t} = (0, 1, s), \quad \frac{\partial X}{\partial s} \times \frac{\partial X}{\partial t} = (-t, -s, 1)$$

(4) continued (why we should wait to calculate)

$$U(s,t) = \frac{1}{\sqrt{1+s^2+t^2}}(-t,-s,1)$$ unit-normal for $z = xy$ given Monge patch.

$$S(v) = -\nabla_v U$$

$$\nabla_{X_s} U = \sum_{i=1}^{3} X_s[\cancel{u_i}]\, U_i$$

just cleaning-up writing.

$U = u_1 U_1 + u_2 U_2 + u_3 U_3$

$$= \sum_{i=1}^{3} X_s[u_i]\, U_i$$

$$= \sum_i \frac{\partial u_i}{\partial x} U_i + \sum_i t\, \frac{\partial u_i}{\partial z} U_i$$

$$\nabla_{X_s} U = X_s\left(\frac{-t}{\sqrt{1+s^2+t^2}}\right) U_1 + X_s\left(\frac{-s}{\sqrt{1+s^2+t^2}}\right) U_2 + X_s\left(\frac{1}{\sqrt{\;}}\right) U_3$$

$X_s = (1, 0, y) \rightarrow \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$

$$g = xy - z$$

$$\nabla g = \langle y, x, -1\rangle$$

$$U = \frac{1}{\sqrt{1+x^2+y^2}}\langle y, x, -1\rangle \rightarrow u_1 = \frac{y}{\sqrt{1+x^2+y^2}}$$

$$u_2 = \frac{x}{\sqrt{1+x^2+y^2}}$$

$$u_3 = \frac{-1}{\sqrt{x^2+y^2+1}}$$

$$\nabla_{X_s} U = X_s[u_1]\, U_1 + X_s[u_2]\, U_2 + X_s[u_3]\, U_3$$

$$= \left(\frac{\partial}{\partial x} + \cancel{y\frac{\partial}{\partial z}}\right)\left(\left(\frac{y}{\sqrt{1+x^2+y^2}}\right) U_1 + \frac{x}{\sqrt{1+x^2+y^2}} U_2 - \frac{1}{\sqrt{x^2+y^2+1}} U_3\right)$$

$$= \frac{-xy\, U_1 - x^2 U_2 + x\, U_3}{\left(\sqrt{1+x^2+y^2}\right)^3}$$

Now do similar for X_t to find $\nabla_{X_t} U$.

§ 5.2 NORMAL CURVATURE

Lemma 2.1: If α is curve in $M \subset \mathbb{R}^3$ then $\alpha'' \cdot U = S(\alpha') \cdot \alpha'$

Proof: Clearly $\underbrace{\alpha' \cdot U = 0}_{\text{along } \alpha}$ as $\alpha' \in T_pM$ whereas $U \in T_pM^{\perp}$.

Differentiate,

$$\alpha'' \cdot U + \alpha' \cdot U' = 0$$

By Method 1, $\nabla_{\alpha'(0)} U = U_\alpha'(0) \Rightarrow S(\alpha') = -\nabla_{\alpha'} U = -U'$

Therefore, $\alpha'' \cdot U = S(\alpha') \cdot \alpha'$. //

Geometric Meaning: $\alpha'' \cdot U$ = component of α'' normal to M. this Lemma says the component depends only on the shape operator and α'. So... ALL CURVES WITH SAME α' WILL HAVE $\alpha'' \cdot U$ same at given point. Rescaling α' to unit-vector $\Rightarrow \alpha'' \cdot U$ measures how M bends in u- direction.

Defn/ Let u be unit tangent to $M \subset \mathbb{R}^3$ at p; $u \in T_pM$ with $\|u\| = 1$. The value

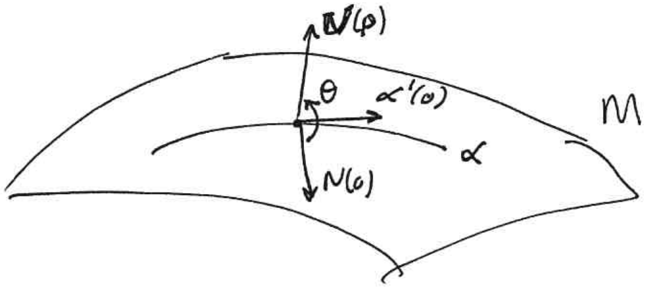
$$k(u) = S(u) \cdot u$$

is the <u>normal curvature</u> of M in the u-direction (at P).

Lemma 2.1: α a curve on $M \subset \mathbb{R}^3$ then $\alpha'' \cdot \mathcal{U} = S(\alpha') \cdot \alpha'$

Proof: Naturally, $\alpha' \cdot \mathcal{U} = 0$ as $\alpha' \in T_pM$ and $\mathcal{U} \in T_pM^{\perp}$.
Hence, $\alpha'' \cdot \mathcal{U} + \alpha' \cdot \mathcal{U}' = 0$. But $S(\alpha') = -\nabla_{\alpha'}\mathcal{U} = -\mathcal{U}_{\alpha}'$.
It follows $\alpha'' \cdot \mathcal{U} = -\alpha' \cdot \mathcal{U}' = \alpha' \cdot S(\alpha')$. (Here I denote $\mathcal{U}'$ to give short hand for $(\mathcal{U} \circ \alpha)'$ aka $\mathcal{U}_{\alpha}'$). //

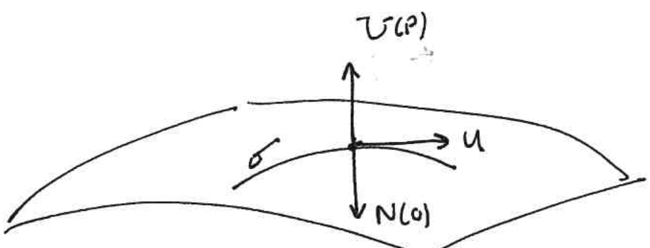
Defn (2.2) Let $u \in T_pM$ with $\|u\| = 1$. The normal curvature of M in the u-direction at p is $k(u) = S(u) \cdot u$



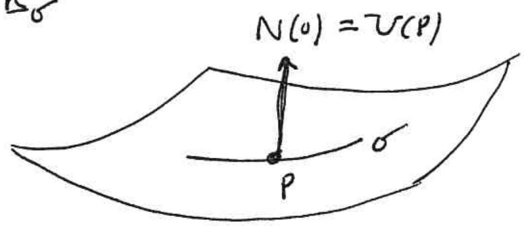
$$
\begin{aligned}
k(u) &= S(u) \cdot u \\
&= \alpha''(0) \cdot \mathcal{U}(P) \\
&= \kappa(0)\, N(0) \cdot \mathcal{U}(P) \\
&= \kappa(0) \cos\theta
\end{aligned}
$$

Given $u \in T_pM$, $\|u\| = 1$ can choose curve σ formed by intersection of plane $\underbrace{\text{span}\{\mathcal{U}(P), u\}}_{\text{point-set}}$ and M: the "normal-section" σ

$$k(u) = \kappa_{\sigma}(0)\, N(0) \cdot \mathcal{U}(P) = \pm \kappa_{\sigma}$$



$k(u) < 0$



$k(u) > 0$

Def^n/(2.3) Let $p \in M \subset \mathbb{R}^3$. The max/min values of $k(u)$ of M at p are called the <u>principal curvatures</u> of M at p and are denoted by k_1, k_2. The directions in which these extreme values occur are called <u>principal directions</u> of M at p. Unit vectors in these directions are called <u>principal vectors</u>.

Def^n/(2.4) A point p of M is <u>umbillic</u> if $k(u)$ is constant for all unit tangent vectors to M at p.

<u>Example</u>: Σ : $x^2+y^2+z^2=R^2 \hookrightarrow k_1 = k_2 = -1/r$ all points umbillic to sphere. Also, a plane is everywhere umbillic.

Th^m/(2.5) (1.) If p is umbillic then the shape operator is just $S(v) = kv$ where $k_1 = k_2 = k$.

(2.) If p is non-umbillic then $k_1 \neq k_2$ (by def^n) and $\exists$ exactly <u>two</u> principal directions, and, there are orthogonal. Furthermore, if e_1, e_2 are principal vectors, then

$$S(e_1) = k_1 e_1 \quad , \quad S(e_2) = k e_2$$

<u>Reinterpretation of Th^m</u>: principal curvatures are eigenvalues and the principal unit directions are eigenvectors of length one for S.

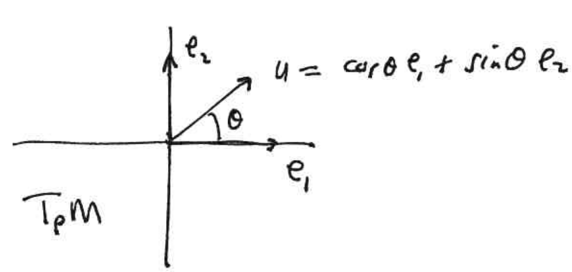
<u>Proof</u>: Suppose $k_{max} = k_1$ in e_1 direction at p, $\|e_1\| = 1$.

$$k_1 = k(e_1) = S(e_1) \cdot e_1$$

Let e_2 be unit vector s.t. $e_1 \cdot e_2 = 0$ and $e_2 \in T_pM$.
Observe $u \in T_pM$ with $\|u\| = 1$ can be written as

$$u = \cos\theta\, e_1 + \sin\theta\, e_2 \quad \text{for some } \theta$$

Thus $k(u) = S(u) \cdot u = S(\cos\theta\, e_1 + \sin\theta\, e_2) \cdot (\cos\theta\, e_1 + \sin\theta\, e_2) = f(\theta)$
that is $k(u) = f(\theta)$, it is a function on $\mathbb{R}$, given our choices for e_1, e_2..



Proof continued: (8)

Let $\mathcal{S}'_{ij} = \mathcal{S}'(e_i)\cdot e_j = \mathcal{S}'(e_j)\cdot e_i = \mathcal{S}'_{ji}$ by symmetry Lemma 1.4 (currently unsubstantiated)

Notice

$$h(\theta) = \mathcal{S}'(\cos\theta e_1 + \sin\theta e_2)\cdot(\cos\theta e_1 + \sin\theta e_2)$$

$$= \cos^2\theta\, \mathcal{S}'(e_1)\cdot e_1 + 2\cos\theta\sin\theta\, \mathcal{S}'(e_1)\cdot e_2 + \sin^2\theta\, \mathcal{S}'(e_2)\cdot e_2$$

$$= \underbrace{\cos^2\theta\, \mathcal{S}'_{11} + 2\cos\theta\sin\theta\, \mathcal{S}'_{12} + \sin^2\theta\, \mathcal{S}'_{22}}_{\hookrightarrow\, k(\theta) = \cos^2\theta\, h_1 + \sin^2\theta\, \mathcal{S}'_{22}} \qquad (*)$$

Find critical θ,

$$\frac{dh}{d\theta} = -2\cos\theta\sin\theta\, \mathcal{S}'_{11} + 2(-\sin^2\theta + \cos^2\theta)\, \mathcal{S}'_{12} + 2\sin\theta\cos\theta\, \mathcal{S}'_{22}$$

$$= 2\sin\theta\cos\theta\left(\mathcal{S}'_{22} - \mathcal{S}'_{11}\right) + 2(\cos^2\theta - \sin^2\theta)\, \mathcal{S}'_{12}$$

Observe, $\theta = 0$ yields $u(0) = e_1$ and we assumed e_1 is principal vector in direction of <u>max</u> $h = h_1$. Fermat's Th$^{\underline{m}}$,

$$\frac{dh}{d\theta}(0) = 0 = 2\mathcal{S}'_{12} \Rightarrow \underline{\mathcal{S}'_{12} = \mathcal{S}'_{21} = 0}.$$

Furthermore, returning to $(*)$ with this newfound insight,

$$h(\theta) = \cos^2\theta\, \mathcal{S}'_{11} + \sin^2\theta\, \mathcal{S}'_{22}$$

$$\mathcal{S}'(e_1) = \mathcal{S}'_{11} e_1, \quad \mathcal{S}'(e_2) = \mathcal{S}'_{22} e_2$$

matrix of $\mathcal{S}'$ w.r.t. $\beta = \{e_1, e_2\}$ basis.

If p is umbillic then

$$\mathcal{S}'(e_1)\cdot e_1 = \mathcal{S}'(e_2)\cdot e_2 = h \Rightarrow [\mathcal{S}']_\beta = \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix}.$$

If p is not umbillic,

$$\mathcal{S}'(e_1)\cdot e_1 = k_1$$
$$\mathcal{S}'(e_2)\cdot e_2 = k_2$$

$\leftarrow$ See next page on why $\mathcal{S}'_{22} = k_2$.

$$[\mathcal{S}']_\beta = \left[\, [\mathcal{S}'(e_1)]_\beta \,\middle|\, [\mathcal{S}'(e_2)]_\beta \,\right]$$

$$= \left[\, [k_1 e_1]_\beta \,\middle|\, [k_2 e_2]_\beta \,\right]$$

$$= \left[\, k_1\begin{pmatrix}1\\0\end{pmatrix} \,\middle|\, k_2\begin{pmatrix}0\\1\end{pmatrix} \right] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}. //$$

continued proof of Th$^{\underline{m}}$ 2.5

$$h(\theta) = c^2 h_1 + s^2 S'_{22} \qquad \text{where} \quad \begin{array}{l} c = \cos\theta \\ s = \sin\theta \end{array}$$

Observe $S'_{22} < h_1$ otherwise $\rightarrow\leftarrow$ $h_1 = h_{max}$. $h_1 = \max h$

Furthermore $h(\theta) = h_1$ iff $c = \pm 1$ and $s = 0$

$\Rightarrow \theta = 0, \pi \Rightarrow e_1$ - direction. Oh, more to the point, h_{min} is S'_{22} obtained for $c = 0$, $s = \pm 1 \Rightarrow e_2$ - direction.

But h_2 is by <u>definition</u> the min. h $\therefore$ $S'_{22} = h_2$ and it follows (as I detailed assuming $S'(e_2) \cdot e_2 = h_2$)

$$[S']_\beta = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}.$$

<u>Remark</u>: the quadratic approximation idea was discussed in Math 332 or 321 notes in some depth... O'neil's nice here, but be careful to understand "x, y" are being used very loosely.

$$K\, V \times W = S(V) \times S(W)$$

$$2H\, V \times W = S(V) \times W + V \times S(W)$$

We calculate,

$$\underbrace{K \, \|V \times W\|^2 = (S(V) \times S(W)) \cdot (V \times W)}$$

Lagrange's Identity, $K = \dfrac{\det \begin{bmatrix} S(V)\cdot V & S(V)\cdot W \\ S(W)\cdot V & S(W)\cdot W \end{bmatrix}}{\det \begin{bmatrix} V\cdot V & V\cdot W \\ V\cdot W & W\cdot W \end{bmatrix}}$

Likewise,

$$2H \, \|V \times W\|^2 = (S(V) \times W)\cdot (V \times W) + (V \times S(W)) \cdot (V \times W)$$

$$H = \frac{\det \begin{bmatrix} S(V)\cdot V & S(V)\cdot W \\ W\cdot V & W\cdot W \end{bmatrix} + \det \begin{bmatrix} V\cdot V & V\cdot W \\ S(W)\cdot V & S(W)\cdot W \end{bmatrix}}{2\det \begin{bmatrix} V\cdot V & W\cdot V \\ W\cdot V & W\cdot W \end{bmatrix}}$$

<u>Remark</u>: these give us formulas for H & K once given <u>any</u> LI set of vectors & the shape op.'s value on said basis at p.

$$\boxed{\text{Cor 3.5} \quad k_1, k_2 = H \pm \sqrt{H^2 - K}}$$

<u>Proof</u>: $K = k_1 k_2$, $H = \dfrac{k_1 + k_2}{2}$ and note

$$H^2 - K = \frac{(k_1 + k_2)^2}{4} - k_1 k_2 = \frac{k_1^2 - 2k_1k_2 + k_2^2}{4} = \left(\frac{k_1 - k_2}{2}\right)^2 \quad //$$

$\boxed{\text{Def}^{\underline{n}} / \text{ A surface } M \subset \mathbb{R}^3 \text{ is } \underline{\text{flat}} \text{ provided } K = 0 \text{ and } \underline{\text{minimal}} \text{ provided } H = 0.}$

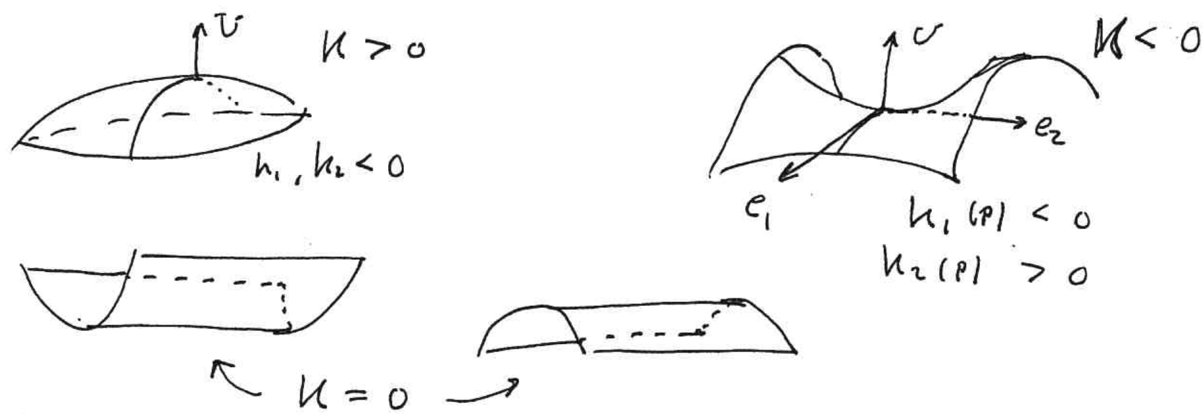
§5.3 GAUSSIAN CURVATURE

Def$^{\underline{n}}$/ GAUSSIAN CURVATURE of M at p is $K = \det S_p'$

Def$^{\underline{n}}$/ MEAN CURVATURE of M at p is $H = \frac{1}{2}\,\text{trace}(S_p')$

Lemma 3.2 $K = k_1 k_2$, $H = \frac{1}{2}(k_1 + k_2)$

Proof: Th$^{\underline{m}}$ 2.5 shows k_1, k_2 are e-values of S_p'.
Linear algebra $\Rightarrow$ $\det(S_p') = k_1 k_2$ & $\text{Trace}(S_p') = k_1 + k_2$ //



Lemma 3.4: If $\{v, w\}$ are LI at $p \in M \subset \mathbb{R}^3$ then,

$$S(v) \times S(w) = K(p)\, v \times w$$
$$S(v) \times w + v \times S(w) = 2H(p)\, v \times w$$

Proof: Let $\beta = \{v, w\}$ and let $[S]_\beta = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Leftrightarrow \begin{array}{l} S(v) = av + bw \\ S(w) = cv + dw \end{array}$

Calculate,

$$S(v) \times S(w) = (av + bw) \times (cv + dw) = (ad - bc)\, v \times w = K\, v \times w.$$

$$\begin{aligned} S(v) \times w + v \times S(w) &= (av + bw) \times w + v \times (cv + dw) \\ &= (a + d)\, v \times w \\ &= \text{Trace}(S)\, v \times w \\ &= 2H\, v \times w. // \end{aligned}$$

§ 5.4 COMPUTATIONAL TECHNIQUES:

- How to calculate K and H via patches? WARPING FUNCTIONS

$$E = \mathbf{x}_u \cdot \mathbf{x}_u \quad , \quad F = \mathbf{x}_u \cdot \mathbf{x}_v \quad , \quad G = \mathbf{x}_v \cdot \mathbf{x}_v$$

Coordinate angle θ

$$F = \mathbf{x}_u \cdot \mathbf{x}_v = \|\mathbf{x}_u\| \, \|\mathbf{x}_v\| \cos\theta = \sqrt{EG}\cos\theta .$$

$$\|\mathbf{x}_u \times \mathbf{x}_v\|^2 = EG - F^2$$

Unit - Normal

$$U = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

Covariant Derivatives amount to $\partial/\partial u$, $\partial/\partial v$ in context ...

$$\mathbf{x}_{uu}, \quad \mathbf{x}_{uv} = \mathbf{x}_{vu}, \quad \mathbf{x}_{vv}$$

Shape operator given by L, M, N:

$$L = S(\mathbf{x}_u) \cdot \mathbf{x}_u$$
$$M = S(\mathbf{x}_u) \cdot \mathbf{x}_v$$
$$N = S(\mathbf{x}_v) \cdot \mathbf{x}_v$$

It follows (work it out!)

$$K = \frac{LN - M^2}{EG - F^2} \quad , \quad H = \frac{GL + EN - 2FM}{2(EG - F^2)}$$

Proof: apply $V = \mathbf{x}_u$, $W = \mathbf{x}_v$ to earlier work. (pg. 220)

§5.5 : Implicit Case

If $M : g = 0$ then $\nabla g = \sum_{i=1}^{3} \frac{\partial g}{\partial x_i} U_i$ is normal

Generally, if $Z(p) \in T_pM^{\perp}$ $\forall p \in M$ then Z is normal vector field and we can form unit-normal by use of Z

$$U = \frac{1}{\|Z\|} Z$$

Let $Z = \sum_i z_i U_i$ we calculate

$$\nabla_V Z = \sum_i V[z_i] U_i$$

Moreover,

$$\nabla_V U = \nabla_V \left(\frac{Z}{\|Z\|} \right) = \frac{1}{\|Z\|} \nabla_V Z + \underbrace{V\left[\frac{1}{\|Z\|} \right] Z}_{-N_V}$$

$$\therefore \quad \underline{S_V U = -\nabla_V U = \frac{-\nabla_V Z}{\|Z\|} + N_V} \quad //$$

Remark: I'm more or less skipping this section, interesting but not to our central story line for this course.

§5.6 SPECIAL CURVES TO A SURFACE $M \subset \mathbb{R}^3$

$\boxed{\text{Def}^{n} (6.1)\ \text{A regular curve } \alpha \text{ in } M \text{ is a principal curve provided the velocity } \alpha' \text{ always points in principal direction}}$

$\boxed{\text{Def}^{n} (6.5)\ \text{A regular curve } \alpha \text{ in } M \text{ is an asymptotic curve provided its velocity } \alpha' \text{ always points in an asymptotic direction}}$

– (asymptotic directions are those for which the normal curvature is <u>zero</u>; $k(u) = S(u) \cdot u$, see §5.2 if forgot) –

$\boxed{\text{Def}^{n} (6.8)\ \text{A curve } \alpha \text{ in } M \subset \mathbb{R}^3 \text{ is a geodesic of } M \text{ provided its acceleration } \alpha'' \text{ is always normal to } M.}$

<u>The essentials summarized: (pg. 247)</u>

Principal curves: $k(\alpha') = k_1$ or k_2, $S(\alpha')$ colinear to α'

Asymptotic curves: $k(\alpha') = 0$, $S(\alpha') \perp$ to α', α'' tangent to M

Geodesic curves: α'' normal to M (nothing about k)

<u>Lemma (6.2) α reg., U the unit normal to M</u>

(1.) α principal iff U' and α' are colinear at each $p \in M$.

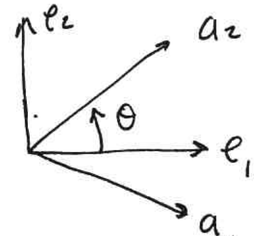
(2.) α is principal curve then the principal curvature of M in the α'-direction is $k_{1,2} = \dfrac{\alpha'' \cdot U}{\alpha' \cdot \alpha'}$.

<u>Lemma (6.3)</u> Let α be curve formed by intersection $M \subset \mathbb{R}^3$ by plane P. If the angle between M and P is constant along α then α is a principal curve of M.

Lemma (6.4)

(1.) If $K(P) > 0$ then $\nexists$ asymptotic directions at P.

(2.) If $K(P) < 0$ then $\exists$ two asymptotic directions at P and these are bisected by principal directions at angle θ s.t. $\tan^2\theta = \dfrac{-k_1(P)}{k_2(P)}$



(3.) If $K(P) = 0$ then every direction is asymptotic at P if P is a planar point, otherwise $\exists$ exactly one asymptotic direction and it's also principal.

Proof: all from Euler's formula $k(u) = k_1(P)\cos^2\theta + k_2(P)\sin^2\theta$.

Remark (pg. 244) a surface is minimal iff through each point $\exists!$ two asymptotic curves which cross $\perp$.

§ 5.7 SURFACES OF REVOLUTION

State Lemma 7-3 (pg. 256)
work out example 7.4