Chapter 6: Geometry of Surfaces in $\mathbb{R}^3$

Def$^{\underline{n}}$ (1.1) (p. 264) An adapted frame field E_1, E_2, E_3 on a region $\mathcal{O}$ in $M \subset \mathbb{R}^3$ is a Euclidean frame field such that E_3 is always normal to M (E_1, E_2 are tangent to M)

Lemma 1.2: $\exists$ an adapted frame field on $\mathcal{O} \subseteq M \subset \mathbb{R}^3$ iff $\mathcal{O}$ is orientable and $\exists$ a nonvanishing vector field V on $\mathcal{O}$.

Proof: orientable $\Rightarrow$ $\exists\, U$, unit-normal on set

Assume orientable and $\exists\, V \neq 0$ on $\mathcal{O}$ note

$E_1 = \frac{V}{\|V\|}$, $E_2 = U$, $E_3 = E_1 \times E_2$

is frame field on $\mathcal{O}$. Conversely if $\exists\, E_1, E_2, E_3$ then can use $E_1 = V$ and $E_3 = U$. //

Examples

(1) Cylinder $x^2 + y^2 = r^2$

$$g = x^2 + y^2 \hookrightarrow \frac{\nabla g}{\|\nabla g\|} = \frac{\langle 2x, 2y, 0\rangle}{2\sqrt{x^2+y^2}} = \underbrace{\frac{x}{r} U_1 + \frac{y}{r} U_2}_{E_3}$$

Next pick U_3 as $E_3 \cdot U_3 = 0$ clearly.

Set $U_3 = E_1$, $\cancel{E_2 \times E_3 = E_1}$ (following text)

$$E_2 = E_3 \times E_1 = \left(\frac{x}{r} U_1 + \frac{y}{r} U_2\right) \times U_3 = \underline{\frac{-x}{r} U_2 + \frac{y}{r} U_1}.$$

Summary,

$$\boxed{\begin{aligned} E_1 &= U_3 \\ E_2 &= \tfrac{1}{r}(-y U_1 + x U_2) \\ E_3 &= \tfrac{1}{r}(x U_1 + y U_2) \end{aligned}}$$

Example Sphere:

$\Sigma : x^2+y^2+z^2 = r^2.$

$$\left.\begin{aligned} E_3 &= \frac{1}{r}\left(xU_1 + yU_2 + zU_3\right) \\ E_2 &= E_3 \times E_1 \\ E_1 &= \frac{V}{\|V\|} \quad \text{where} \quad V = -yU_1 + xU_2 \end{aligned}\right\}$$

I call these $\hat{\rho}, \hat{\theta}, \hat{\varphi}$ in my calculus III notes.

Covers Σ modulo N & S poles $(0,0,\pm r)$.

$$E_1 = \frac{1}{\sqrt{x^2+y^2}}\left(-yU_1 + xU_2\right)$$

$$E_2 = \frac{1}{r\sqrt{x^2+y^2}}\left(xU_1 + yU_2 + zU_3\right) \times \left(-yU_1 + xU_2\right).$$

$$= \frac{1}{r\sqrt{x^2+y^2}}\left(x^2U_3 + y^2U_3 - yzU_2 - xzU_1\right)$$

$$= \frac{1}{r\sqrt{x^2+y^2}}\Big(\left(x^2+y^2\right)U_3 - z\left(xU_1 + yU_2\right)\Big) = E_2.$$

$$\|E_2\|^2 = \frac{1}{r^2(x^2+y^2)}\left((x^2+y^2)^2 + z^2x^2 + z^2y^2\right)$$

$$= \frac{1}{r^2(x^2+y^2)}\left((x^2+y^2)\left[x^2+y^2+z^2\right]\right)$$

$$= \frac{r^2}{r^2}$$

$$= 1. \quad \text{good.}$$

$$E_3 = \frac{1}{r}\left(xU_1 + yU_2 + zU_3\right).$$

If we imagine expanding M along E_3 then we obtain a three dimensional open set on which the §2.7 – 2.8 eq^{n}'s apply: Just extend E_1, E_2, E_3 on M to points near-by M.

$$\left(\nabla_v E_i\right)(p) = \sum \omega_{ij}(v)\, E_j(p)$$

Th^{m} (1.4) E_1, E_2, E_3 an adapted frame on $M \subset \mathbb{R}^3$ and $v \in T_pM$,

$$\nabla_v E_i = \sum_{j=1}^{3} \omega_{ij}(v)\, E_j(p)$$

<u>Concept</u>: $\omega_{ij}(v)$ measures initial rate at which E_i rotates towards E_j as we move from p in the v-direction.

<u>Remark</u>: $S(v) = -\nabla_v U$ but $U = E_3$ here so the connection forms reveal the details of the shape operator.

<u>Cor (1.5)</u> Let S be shape operator gotten from E_3 and E_1, E_2, E_3 an adapted frame to M. Then for each $v \in T_pM$,

$$S(v) = \omega_{13}(v)\, E_1(p) + \omega_{23}(v)\, E_2(p)$$

<u>Proof</u>:
$$\begin{aligned} S(v) &= -\nabla_v E_3 \\ &= -\omega_{31}(v)\, E_1 - \omega_{32}(v)\, E_2 \\ &= \underline{\omega_{13}(v)\, E_1 + \omega_{23}(v)\, E_2}. \end{aligned}$$

Dual Forms to E_1, E_2, E_3 are $\theta_1, \theta_2, \theta_3$
and $\theta_i(v) = v \cdot E_i(p) \iff \theta_i(E_j) = \delta_{ij}$.
However, $\theta_3 \big|_{T_pM} \equiv 0$. Then, on the surface M
we have just

$$\theta_1, \theta_2 \quad \text{dual to } E_1, E_2$$
$$\omega_{12} \quad \text{rate of } E_1 \text{ rotating to } E_2$$
$$\omega_{13}, \omega_{23} \quad \text{describe shape operator tied to } E_3$$

<u>Example</u>: the sphere modulo(N/S) poles has frame
E_1, E_2, E_3 as on ② of these notes. In spherical
coordinates the dual forms are much nicer!

$$\theta_1 = r\cos\varphi \, d\theta \qquad \omega_{12} = \sin\varphi \, d\theta$$
$$\theta_2 = r\,d\varphi \qquad \omega_{13} = -\cos\varphi \, d\theta$$
$$\omega_{23} = -d\varphi$$

The $\rho = \sqrt{x^2+y^2+z^2} = r$ on sphere $\Sigma_r = M$.

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$$\begin{cases} F_1 = \cos\varphi(\cos\theta\, U_1 + \sin\theta\, U_2) + \sin\varphi\, U_3 \\ F_2 = -\sin\theta\, U_1 + \cos\theta\, U_2 \\ F_3 = -\sin\varphi(\cos\theta\, U_1 + \sin\theta\, U_2) + \cos\varphi\, U_3 \end{cases}$$

He says $1 \to 3$, $2 \to 1$, $3 \to 2$ to
translate Chpt. 2 to Chpt. 6.

Th^{m} (1.7) If E_1, E_2, E_3 is adapted to $M \subset \mathbb{R}^3$ and θ_1, θ_2 are dual to E_1, E_2 then, (5)

$$(1.)\ \begin{cases} d\theta_1 = \omega_{12} \wedge \theta_2 \\ d\theta_2 = \omega_{21} \wedge \theta_1 \end{cases}$$ 1st Structural Eq^{n}'s.

$$(2)\quad \omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2 = 0$$ } symmetry eq^{n}

$$(3.)\quad d\omega_{12} = \omega_{13} \wedge \omega_{32}$$ Gauss Eq^{n}.

$$(4)\ \begin{cases} d\omega_{13} = \omega_{12} \wedge \omega_{23} \\ d\omega_{23} = \omega_{21} \wedge \omega_{13} \end{cases}$$ Codazzi Eq^{n}'s. describe change in shape from point to point.

<u>Proof</u>: apply Cartan's Structure Eq^{n}'s.

$$d\theta = \omega \wedge \theta \quad \& \quad d\omega = \omega \wedge \omega.$$

§ 6.2 Form Computations

$$V = (V \cdot E_1) E_1 + (V \cdot E_2) E_2$$

<u>1-form</u>: $\phi = \phi(E_1)\theta_1 + \phi(E_2)\theta_2$ } Lemma 2.1

<u>2-form</u>: $\psi = \psi(E_1, E_2)\, \theta_1 \wedge \theta_2$

<u>Lemma 2.2</u>: (1) $\omega_{13} \wedge \omega_{23} = K\, \theta_1 \wedge \theta_2$

(2.) $\omega_{13} \wedge \theta_2 + \theta_1 \wedge \omega_{23} = 2H\, \theta_1 \wedge \theta_2$

<u>Proof</u>: $[S]_{\beta,\beta} = \big[[S(E_1)]_\beta \,\big|\, [S(E_2)]_\beta \big] \qquad \beta = \{E_1, E_2\}$

$$S(E_1) = -\nabla_{E_1} E_3 = -\omega_{31}(E_1)E_1 - \omega_{32}(E_1)E_2 \qquad \nabla_V E_i = \sum_{j=1}^{3} \omega_{ij}(V)\, E_j$$

$$S(E_2) = -\nabla_{E_2} E_3 = -\omega_{31}(E_2)E_1 - \omega_{32}(E_2)E_2$$

$$[S(E_1)]_\beta = \begin{pmatrix} \omega_{13}(E_1) \\ \omega_{23}(E_1) \end{pmatrix} \quad \& \quad [S(E_2)]_\beta = \begin{pmatrix} \omega_{13}(E_2) \\ \omega_{23}(E_2) \end{pmatrix}$$

$$\therefore [S]_{\beta,\beta} = \begin{bmatrix} \omega_{13}(E_1) & \omega_{13}(E_2) \\ \omega_{23}(E_1) & \omega_{23}(E_2) \end{bmatrix} \hookrightarrow K = \det [S]_{\beta\beta} = \omega_{13}(E_1)\omega_{23}(E_2) - \omega_{13}(E_2)\omega_{23}(E_1)$$

Lemma 2.2 continued

$$\omega_{13} \wedge \omega_{23} = \left(\omega_{13}(E_1)\theta_1 + \omega_{13}(E_2)\theta_2\right) \wedge \left(\omega_{23}(E_1)\theta_1 + \omega_{23}(E_2)\theta_2\right)$$
$$= \left(\omega_{13}(E_1)\omega_{23}(E_2) - \omega_{13}(E_2)\omega_{23}(E_1)\right)\theta_1 \wedge \theta_2$$
$$= \det([S]_\beta)\ \theta_1 \wedge \theta_2$$
$$= K\ \theta_1 \wedge \theta_2 . \qquad (K \text{ Gaussian Curvature})$$

– Behold! Lemma 3.4 in form notation. – (see 219)

$$\omega_{13} \wedge \theta_2 + \theta_1 \wedge \omega_{23} = \left(\omega_{13}(E_1) + \omega_{23}(E_2)\right)\theta_1 \wedge \theta_2$$
$$= \mathrm{Trace}([S]_{\beta,\beta})\ \theta_1 \wedge \theta_2$$
$$= 2H\ \theta_1 \wedge \theta_2 \qquad (H \text{ mean curvature})$$

$$\boxed{\text{Cor (2.3)} \quad d\omega_{12} = -K\theta_1 \wedge \theta_2}$$

Proof: $\underbrace{d\omega_{12} = \omega_{13} \wedge \omega_{32}}_{\text{Gauss' Eq}^{n}\text{ (3) of Th}^{m}\text{ 1.7 or pg. 5 of these notes.}} = -\omega_{13} \wedge \omega_{23} = -K\ \theta_1 \wedge \theta_2$. //

Example: sphere.

$$\theta_1 \wedge \theta_2 = r^2 \cos\phi\ d\theta \wedge d\phi = -r^2 \cos\phi\ d\phi \wedge d\theta$$
$$d\omega_{12} = d(\sin\phi\ d\theta) = d(\sin\phi) \wedge d\theta = \cos\phi\ d\phi \wedge d\theta$$
$$\Rightarrow \boxed{K = 1/r^2}$$
$$d\omega_{12} = \cos\phi\ d\phi \wedge d\theta = -K(-r^2 \cos\phi\ d\phi \wedge d\theta)$$
$$\Rightarrow Kr^2 = 1 \quad \therefore \ \underline{K = 1/r^2}.$$

Def^{n}/ A principal frame field on $M \subset \mathbb{R}^3$ is an adapted frame field E_1, E_2, E_3 s.t. each point E_1 & E_2 are principal vectors of M.

- in the absense of umbillics ($k_1 = k_2$) the principal frame is uniquely determined at each pt. modulo ± 1.

Lemma 2.5: If p is nonumbilic then $\exists$ principal frame field on some neighboorhood of p in M.

Proof: use eigenvector basis for $[S_p]$ at each p and as $k_1 \neq k_2$ near p it follows the eigenvectors of $[S_p]$ give princ. frame. // (see p. 271-272)

Oh, so what, if E_1, E_2, E_3 is principal frame then

$$\omega_{13}(E_1) = k_1 \qquad \omega_{13}(E_2) = 0$$
$$\omega_{23}(E_1) = 0 \qquad \omega_{23}(E_2) = k_2$$

Therefore,

$$\boxed{\omega_{13} = k_1\theta_1 \quad \& \quad \omega_{23} = k_2\theta_2}$$ nice!

Thus, the Coduzzi Eq^{ns},

Th^{m} (2.6)
$$E_1[k_2] = (k_1 - k_2)\,\omega_{12}(E_2)$$
$$E_2[k_1] = (k_1 - k_2)\,\omega_{12}(E_1)$$

k_1 & k_2 are functions on M.

- proof on pg. 272 -

Th^m (2.6) If E_1, E_2, E_3 is principal frame field on $M \subset \mathbb{R}^3$ then $E_1[k_2] = (k_1 - k_2)\,\omega_{12}(E_2)$ and $E_2[k_1] = (k_1 - k_2)\,\omega_{12}(E_1)$

Proof: $S(E_1) = k_1 E_1$ & $S(E_2) = k_2 E_2$

Yet $S(V) = \omega_{13}(V)E_1 + \omega_{23}(V)E_2$ (cor 1-5 pg. 266)

So, $S(E_1) = \omega_{13}(E_1)E_1 + \omega_{23}(E_1)E_2 = k_1 E_1 \Rightarrow \begin{cases}\omega_{23}(E_1) = 0 \\ \omega_{13}(E_1) = k_1\end{cases}$

$S(E_2) = \omega_{13}(E_2)E_1 + \omega_{23}(E_2)E_2 = k_2 E_2$

$\hookrightarrow \omega_{13}(E_2) = 0$, $\omega_{23}(E_2) = k_2$

Hence $\omega_{13} = k_1 \theta_1$ and $\omega_{23} = k_2 \theta_2$ (preamble to Th^m 2.6)

Codazzi Eq^n's.:

$$d\omega_{13} = \omega_{12} \wedge \omega_{23}$$
$$d\omega_{23} = \omega_{21} \wedge \omega_{13}$$

Yet, we can calculate, note $k_1 : M \to \mathbb{R}$ a fnct.

$$d\omega_{13} = dk_1 \wedge \theta_1 + k_1 d\theta_1 = \omega_{12} \wedge \omega_{23}$$
$$d\omega_{23} = \underbrace{dk_2 \wedge \theta_2 + k_2 d\theta_2}_{} = \omega_{21} \wedge \omega_{13}$$

use 1st structure eq^n, $d\theta = \omega \wedge \theta$ $\to$ $d\theta_1 = \omega_{12} \wedge \theta_2$, $d\theta_2 = \omega_{21} \wedge \theta_1$

Hence,

$$dk_1 \wedge \theta_1 + k_1 \omega_{12} \wedge \theta_2 = \omega_{12} \wedge \omega_{23} = k_2 \omega_{12} \wedge \theta_2$$
$$\Rightarrow dk_1 \wedge \theta_1 = (k_2 - k_1)\,\omega_{12} \wedge \theta_2$$

$$(c_1\theta_1 + c_2\theta_2) \wedge \theta_1 = (k_2 - k_1)\,\omega_{12} \wedge \theta_2$$
$$= (k_2 - k_1)(b_1\theta_1 + b_2\theta_2) \wedge \theta_2$$

$$\Rightarrow c_2\,\theta_2 \wedge \theta_1 = (k_2 - k_1)\,b_1\,\theta_1 \wedge \theta_2$$

$$\Rightarrow c_2 = b_1(k_1 - k_2) \quad \therefore dk_1 = b_1(k_1 - k_2)\theta_2$$

§ 6.3 Some Global Th^{m}s

Th^m (3.1) If $S \equiv 0$ then M is part of plane.

~~Th^m~~ Lemma (3.2) If M is an all-umbilic surface in $\mathbb{R}^3$ then M has constant Gaussian curvature $K \geq 0$.

Th^m (3.3) If $M \subset \mathbb{R}^3$ is all umbilic and $K > 0$ then M is part of sphere in $\mathbb{R}^3$ of radius $1/\sqrt{K}$.

Cor. 3.4 A compact all-umbilic surface M in $\mathbb{R}^3$ is an entire sphere.

Th^m 3.5 On every compact surface M in $\mathbb{R}^3$ $\exists$ a point at which $K > 0$. (p. 277)

Lemma 3.6 (Hilbert) Let m be a point of $M \subset \mathbb{R}^3$ such that

(1) k_1 has local max at m

(2) k_2 has local min at m

(3.) $k_1(m) > k_2(m)$

Then $K(m) \leq 0$. (Proof p. 279, nontrivial)

Th^m (3.7) If M is compact surface in $\mathbb{R}^3$ with constant Gauss curv. K then M is a sphere of radius $1/\sqrt{K}$. (Liebmann's Th^m)

We saw in § 5.7 that compactness is necessary, $\exists$ noncompact, constant positive curvature surfaces.

§ 6.4 Isometries & Local Isometries

Def^n/ If $p, q \in M \subset \mathbb{R}^3$ and $S = \{\alpha \mid \alpha$ curve from p to $q\}$ then $\rho(p,q) = \inf\{L(\alpha) \mid \alpha \in S\}$. The <u>intrinsic</u> <u>distance</u> from p to q is so defined.

<u>Comment</u>: the straight-line distance may be shorter! Intrinsic distance requires we move along M.

Def^n (4.2) An isometry $F: M \to \bar{M}$ of surfaces in $\mathbb{R}^3$ is a 1-1 mapping for which $d_pF(V_p) \cdot d_pF(W_p) = V_p \cdot W_p$ $\forall\, V_p, W_p \in T_pM$ and here $d_pF = F_*$ in Oneil.

Observe $d_pF(V_p) = 0 \Rightarrow 0 = d_pF(V_p) \cdot d_pF(V_p) = V_p \cdot V_p = \|V_p\|^2$
thus $\mathrm{Ker}(d_pF) = \{0\}$ $\therefore$ d_pF is injective
$\Rightarrow \exists\, U$ for which $F|_U : U \to F(U)$ is diffeomorphism
thus F is diffeomorphism as it has smooth inverse locally & injectivity of $F \Rightarrow F^{-1}: \bar{M} \to M$ exists and as it matches the local.... (see pg. 169)

Th^m (4.3) ISOMETRIES PRESERVE INTRINSIC DISTANCE.
If $F: M \to \bar{M}$ is an isometry then

$$\rho(p, q) = \bar{\rho}(F(p), F(q))$$

where $\rho, \bar{\rho}$ are intrinsic distance fncts on $M, \bar{M}$ respective.

<u>Proof</u>: essentially this $\bar{\alpha} = F \circ \alpha$ has $L(\bar{\alpha}) = L(\alpha)$ since $\bar{\alpha}' = dF(\alpha')$ and $\bar{\alpha}' \cdot \bar{\alpha}' = \alpha' \cdot \alpha'$.

$\boxed{\text{Def}^{\underline{n}}\ (4.4)\ \text{a \underline{local} \underline{isometry} of } F: M \to N \text{ is a mapping that preserves dot-products of tangent vectors.}}$

- The difference between (global) isometry is there is no expectation of 1-1 for F.
- An isometry is a local isometry which is 1-1 and onto.

Lemma 4.5: Let $F: M \to N$ be a mapping. For each patch $\mathbf{x}: D \to M$ consider the composite map,

$$\bar{\mathbf{x}} = F(\mathbf{x}) : D \to N$$

Then F is local isometry iff for each patch $\mathbf{x}$ we have $E = \bar{E}$, $F = \bar{F}$, $G = \bar{G}$ where $\bar{\mathbf{x}}$ defines $\bar{E}, \bar{F}, \bar{G}$ as usual even though $\bar{\mathbf{x}}$ may not be a patch.

Proof: $E = \mathbf{x}_u \cdot \mathbf{x}_u$, $F = \mathbf{x}_u \cdot \mathbf{x}_v$, $G = \mathbf{x}_v \cdot \mathbf{x}_v$

whereas $\bar{E} = \bar{\mathbf{x}}_u \cdot \bar{\mathbf{x}}_u$, $\bar{F} = \bar{\mathbf{x}}_u \cdot \bar{\mathbf{x}}_v$, $\bar{G} = \bar{\mathbf{x}}_v \cdot \bar{\mathbf{x}}_v$

If F is isometry then $dF(\mathbf{x}_u) \cdot dF(\mathbf{x}_u) = \mathbf{x}_u \cdot \mathbf{x}_u$

$\Rightarrow \bar{E} = E$. Likewise $\bar{F} = F$, $\bar{G} = G$.

Example (4.6)

- Isometry of Plane onto surface
- $\Phi : \mathbb{R}^2 \to M$ a plane has

$$\Phi_*(v_i) \cdot \Phi_*(v_j) = v_i \cdot v_j = \delta_{ij}.$$

$$\Rightarrow \underline{E = 1,\ F = 0,\ G = 1}.$$

- $\Phi : \mathbb{R}^2 \to C \subset \mathbb{R}^3$ cylinder,

$$\Phi(u,v) = \left(r\cos\left(\tfrac{u}{r}\right), r\sin\left(\tfrac{u}{r}\right), v\right)$$

$$\Phi_u = \left(-\sin\left(\tfrac{u}{r}\right), \cos\left(\tfrac{u}{r}\right), 0\right) \hookrightarrow \Phi_u \cdot \Phi_u = 1 = E$$

$$\hookrightarrow \Phi_u \cdot \Phi_v = 0 = F$$

$$\Phi_v = (0, 0, 1) \hookrightarrow \Phi_v \cdot \Phi_v = 1 = G$$

Thus cylinder is isometric to plane, <u>locally</u>.

$\cosh^2 z - \sinh^2 z = 1$

$\sinh(z) = u$

$1 = \cosh(z)\frac{dz}{du}$

$\frac{d}{du}\left[\sinh^{-1}(u)\right] = \frac{1}{\cosh z} = \frac{1}{\sqrt{1+u^2}}$

Example 4.6.2 p. 285

$$\Phi(u,v) = (u\cos v, u\sin v, v)$$

$$\Phi_u = (\cos v, \sin v, 0)$$

$$\Phi_v = (-u\sin v, u\cos v, 1)$$

$$\Phi_u \cdot \Phi_u = 1 = E$$

$$\Phi_v \cdot \Phi_v = u^2 + 1 = G$$

$$F = \Phi_u \cdot \Phi_v = 0.$$

$$\bar{\Phi} = \Psi(u,v) = \left(\sinh^{-1}(u), \sqrt{1+u^2}\cos v, \sqrt{1+u^2}\sin v\right)$$

$$\Psi_u = \left(\frac{1}{\sqrt{1+u^2}}, \frac{u\cos v}{\sqrt{1+u^2}}, \frac{u\sin v}{\sqrt{1+u^2}}\right) \qquad \Psi_u \cdot \Psi_u = \frac{1+u^2}{(\sqrt{1+u^2})^2} = 1 = \bar{E}$$

$$\Psi_v = \left(0, -\sqrt{1+u^2}\sin v, \sqrt{u^2+1}\cos(v)\right) \qquad \Psi_v \cdot \Psi_v = u^2+1 = \bar{G}$$

$$\Psi_u \cdot \Psi_v = 0.$$

<u>Question</u>: $\Psi(u,v) = F(\Phi(u,v))$ for which isometry F?

Defn 4.7 $F : M \to N$ is <u>conformal</u> provided $\exists\, \lambda : M \to \mathbb{R}$, $\lambda(M) > 0$ s.t. $\| dF(v_p) \| = \underline{\lambda(p)} \| v_p \|$ $\forall\, v_p \in T_pM$ and $\forall\, p \in M$.

scale factor, two surfaces conformal iff $E\lambda^2(\Phi) = \bar{E}$ etc...

Remark: conformal geometry has interesting connections with the theory of complex variables. You can search about Catenoid & Heliocoid online and find fascinating connections with complex analysis.

§6.5: INTRINSIC GEOMETRY OF SURFACES IN $\mathbb{R}^3$

- CONCEPT: the intrinsic geometry of $M \subset \mathbb{R}^3$ consists of those concepts – called isometric invariants – that are preserved by all isometries, $F: M \longrightarrow \overline{M}$.

  - intrinsic distance (Th^m 4.3) is an isometric invariant
  - shape operator, k_1, k_2, principal directions, mean curvature do not belong to the intrinsic geometry of M. Instead, they are extrinsic as they pertain to how M is situated in the ambient $\mathbb{R}^3$.

We return to §6.1 and now work towards discovering how our geometry on $M \subset \mathbb{R}^3$ can be defined w/o reference to E_3 which is extrinsic. Our primary tools are θ_1, θ_2 and E_1, E_2 which are intrinsic... can we use these to determine connection form ω_{12} (we already give up on the other forms as you may recall ω_{13}, ω_{23} determine the SHAPE OPERATOR which is extrinsic)

Lemma 5.1: The connection form $\omega_{12} = -\omega_{21}$ is the only 1-form that satisfies $d\theta_1 = \omega_{12} \wedge \theta_2$, $d\theta_2 = \omega_{21} \wedge \theta_1$

$$d\theta_1(E_1, E_2) = E_1(\theta_1(E_2)) - E_2(\theta_1(E_1)) = 0.$$

(false, this formula needs modification for frame. In pg. 161, def$^=$ 4.4 Σ_u, Σ_v are coord. derivations.)

Lemma 5.1: $\omega_{12} = -\omega_{21}$ is the only one form such that $d\theta_1 = \omega_{12} \wedge \theta_2$ and $d\theta_2 = \omega_{21} \wedge \theta_1$

Proof: 1st we observe ω_{12} does solve $d\theta_1 = \omega_{12} \wedge \theta_2$ and $d\theta_2 = \omega_{21} \wedge \theta_1$. Moreover,

$$\omega_{12} = \omega_{12}(E_1)\theta_1 + \omega_{12}(E_2)\theta_2$$

$$\omega_{12} \wedge \theta_2 = \omega_{12}(E_1)\theta_1 \wedge \theta_2 \Rightarrow (\omega_{12} \wedge \theta_2)(E_1, E_2) = \omega_{12}(E_1)$$

Thus, as $d\theta_1 = \omega_{12} \wedge \theta_2 \Rightarrow \underline{d\theta_1(E_1, E_2) = \omega_{12}(E_1)}$. $*$

$$\omega_{21} = \omega_{21}(E_1)\theta_1 + \omega_{21}(E_2)\theta_2$$

$$\omega_{21} \wedge \theta_1 = \omega_{21}(E_2)\theta_2 \wedge \theta_1 = -\omega_{21}(E_2)\theta_1 \wedge \theta_2$$

$$d\theta_2 = \omega_{21} \wedge \theta_1 \Rightarrow \underline{d\theta_2(E_1, E_2) = -\omega_{21}(E_2)}. \; **$$

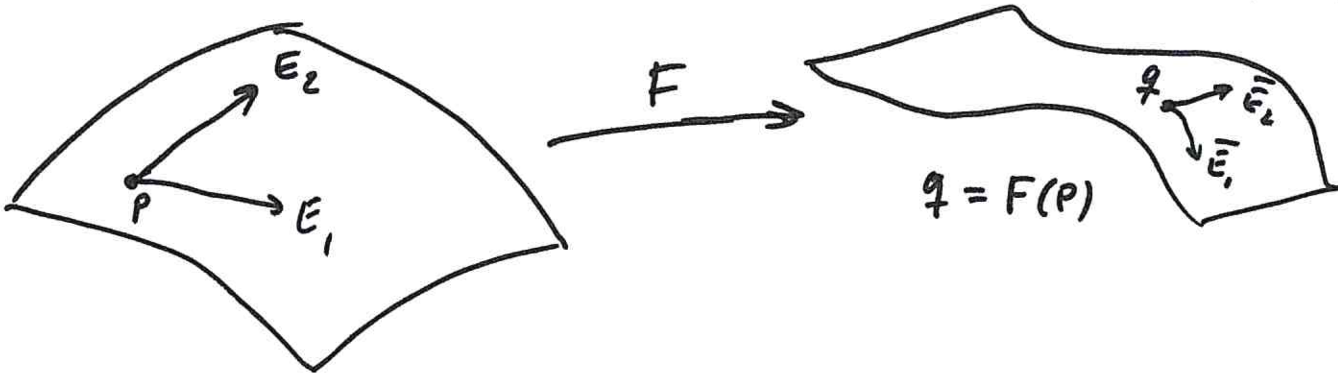
The formulas $*$ and $**$ show $d\theta_1$ & $d\theta_2$ uniquely determine the one-form $\omega_{12} = \omega_{12}(E_1)\theta_1 + \omega_{12}(E_2)\theta_2$

Def$^{\underline{n}}$/ Intrinsic $\omega_{12}(V) = \omega_{12}(v_1 E_1 + v_2 E_2)$
$= v_1\, \omega_{12}(E_1) + v_2\, \omega_{12}(E_2)$

where $\omega_{12}(E_1) = d\theta_1(E_1, E_2)$ by def$^{\underline{n}}$.
and $\omega_{12}(E_2) = d\theta_2(E_1, E_2)$ by def$^{\underline{n}}$.

$$\boxed{\omega_{12}(v_1 E_1 + v_2 E_2) = v_1\, d\theta_1(E_1, E_2) + v_2\, d\theta_2(E_1, E_2)}$$

$F : M \longrightarrow \bar{M}$ an isometry



$$\bar{E}_1(q) = F_*(E_1(p)) \longrightarrow \bar{E}_1 = F_*(E_1)$$
$$\bar{E}_2(q) = F_*(E_2(p)) \longrightarrow \bar{E}_2 = F_*(E_2)$$

$$\bar{E}_j \cdot \bar{E}_i = F_*(E_j) \cdot F_*(E_i) = E_j \cdot E_i = \delta_{ji}$$

Likewise $\| \bar{E}_i \| = \| F_*(E_i) \| = \| E_i \| = 1.$

Thus $\{E_1, E_2\}$ frame $\Rightarrow$ $\{\bar{E}_1, \bar{E}_2\}$ a frame.

Lemma 5.3: Let $F : M \longrightarrow \bar{M}$ be an isometry and let E_1, E_2 be a frame field on M. If $\bar{E}_1, \bar{E}_2$ is the frame field on $\bar{M}$ transferred by F_* then,

$$\left.\begin{aligned} &(1.)\quad \theta_1 = F^*(\bar{\theta}_1) \\ &\quad \theta_2 = F^*(\bar{\theta}_2) \end{aligned}\right\} \text{pull-backs.}$$

$$(2)\quad \omega_{12} = F^*(\bar{\omega}_{12})$$

<u>Proof</u>: Let $\bar{\theta}_1 = F_*(\theta_1)$ and $\bar{\theta}_2 = F_*(\theta_2)$

$$\left(F^*(\bar{\theta}_j)\right)(E_i) = \bar{\theta}_j\left(F_*(E_i)\right) = \bar{\theta}_j\left(\bar{E}_i\right) = \delta_{ji}$$

$$\therefore \quad \underline{F^*(\bar{\theta}_j) = \theta_j} . \quad *$$

$$* \quad \theta_j = \theta_j(E_1)\theta_1 + \theta_j(E_2)\theta_2 \Rightarrow \theta_j(E_i) = \delta_{ji}$$
$$\Rightarrow F^*(\bar{\theta}_j) = \theta_j .$$

The proof of (1.) is just def^{n} of pull-back applied to $\bar{\theta}_1$, $\bar{\theta}_2$. Next,

$$\left(F^*(\bar{\omega}_{12})\right)(E_1, E_2) = \bar{\omega}_{12}\left(F_*(E_1), F_*(E_2)\right)$$
$$= \bar{\omega}_{12}\left(\bar{E}_1, \bar{E}_2\right)$$

But, the δ_{ij} is not shared by two-forms .... need another approach.

$$d\bar{\theta}_1 = \bar{\omega}_{12} \wedge \bar{\theta}_2 \quad \text{on } \bar{M}$$
$$d(F^*\bar{\theta}_1) = F^*(d\bar{\theta}_1) = F^*(\bar{\omega}_{12} \wedge \bar{\theta}_2)$$
$$= F^*(\bar{\omega}_{12}) \wedge F^*(\bar{\theta}_2)$$

$$\therefore d\theta_1 = F^*(\bar{\omega}_{12}) \wedge \theta_2$$

Similarly $\quad d\theta_2 = F^*(\bar{\omega}_{21}) \wedge \theta_1 = -F^*(\bar{\omega}_{12}) \wedge \theta_1$

Thus $\quad \omega_{12} = F^*(\bar{\omega}_{21})$ and $\omega_{21} = F^*(\bar{\omega}_{21})$

$\boxed{\begin{array}{l}\text{Th}^{\underline{m}}\text{(5.4) GAUSSIAN CURVATURE IS AN ISOMETRIC INVARIANT. THAT IS} \\ \text{IF } F: M \longrightarrow \bar{M} \text{ an isometry then } K(p) = \bar{K}(F(p)). \\ \forall p \in M.\end{array}}$

<u>Proof</u>: Lemma 5.3 gives $F^*(\bar{\theta}_i) = \theta_i$ and $F^*(\bar{\omega}_{ij}) = \omega_{ij}$ for frame field E_1, E_2 on M and transferred $\bar{E}_1, \bar{E}_2$ on $\bar{M}$ where $\bar{E}_j = F_*(E_j)$ for $j = 1, 2$. Recall

$$d\bar{\omega}_{12} = -\bar{K}\,\bar{\theta}_1 \wedge \bar{\theta}_2 \quad \text{(previous Lecture)}$$

$$d\omega_{12} = -K\,\theta_1 \wedge \theta_2$$

Observe,

$$\begin{aligned} F^*(d\bar{\omega}_{12}) &= -F^*(\bar{K}\,\bar{\theta}_1 \wedge \bar{\theta}_2) \\ &= -F^*(\bar{K})\,F^*(\bar{\theta}_1) \wedge F^*(\bar{\theta}_2) \\ &= -(\bar{K} \circ F)\,\theta_1 \wedge \theta_2 \end{aligned}$$

$$d(F^*\bar{\omega}_{12}) = d\omega_{12} = -K\,\theta_1 \wedge \theta_2$$

Thus, $K = \bar{K}(F)$. //