§6.6 ORTHOGONAL COORDINATES

The central eq^{n}s of last section,

$$d\theta_1 = \omega_{12} \wedge \theta_2$$
$$d\theta_2 = \omega_{21} \wedge \theta_1$$
$$d\omega_{12} = -K\,\theta_1 \wedge \theta_2$$

Now we develop a technique for intrinsic calculation of K.

Def^{n} An orthogonal coordinate path $\mathbf{X} : D \longrightarrow M$ is one for which $F = \mathbf{X}_u \cdot \mathbf{X}_v = 0$. Moreover, the associated frame field of an orthogonal patch $\mathbf{X} : D \longrightarrow M$ consists of E_1, E_2 on $\mathbf{X}(D)$ defined by

$$E_1 = \frac{\mathbf{X}_u(u,v)}{\sqrt{E}(u,v)} \qquad E_2 = \frac{\mathbf{X}_v(u,v)}{\sqrt{G}(u,v)}$$

or in my usual calc III. notation, $E_1 = \widehat{\mathbf{X}_u}$, $E_2 = \widehat{\mathbf{X}_v}$

We now introduce some notation at odds with the notation of the last 2 or 3 chapters.

$$U : \mathbf{X}(D) \longrightarrow D$$
$$V : \mathbf{X}(D) \longrightarrow D$$

Technically, $(u,v) = \mathbf{X}^{-1}$ and we had used $\tilde{u}, \tilde{v}$ to denote these previously. Anyway,

$$E_1 = \frac{\mathbf{X}_u}{\sqrt{E}}, \quad E_2 = \frac{\mathbf{X}_v}{\sqrt{G}}$$

dual forms θ_1, θ_2 have $\theta_i(E_j) = \delta_{ij}$

$$\left.\begin{array}{ll} du(\mathbf{X}_u) = 1 & dV\cdot(\mathbf{X}_u) = 0 \\ du(\mathbf{X}_v) = 0 & dV(\mathbf{X}_v) = 1 \end{array}\right\}$$ see Ex. 7 of §4.4 with $\tilde{u}, \tilde{v}$ etc...

$$du(\mathbf{X}_u) = 1 \qquad dv(\mathbf{X}_u) = 0$$
$$du(\mathbf{X}_v) = 0 \qquad dv(\mathbf{X}_v) = 0$$

$$E_1 = \frac{\mathbf{X}_u}{\sqrt{E}} \qquad E_2 = \frac{\mathbf{X}_v}{\sqrt{G}} \qquad \begin{array}{l} E = \mathbf{X}_u \cdot \mathbf{X}_u \\ G = \mathbf{X}_v \cdot \mathbf{X}_v \end{array}$$

<u>Claim</u>: $\theta_1 = \sqrt{E}\, du$, $\theta_2 = \sqrt{G}\, dv$ for orthogonal coordinates (u,v) on M.

<u>Proof</u>: $E_1(\theta_1) = \frac{1}{\sqrt{E}} \mathbf{X}_u(\sqrt{E}\, du) = \underbrace{\frac{1}{\sqrt{E}}\sqrt{E}}_{1}\, \underbrace{du(\mathbf{X}_u)}_{1} = 1.$

$$E_1(\theta_2) = \frac{1}{\sqrt{E}} \mathbf{X}_u(\sqrt{G}\, dv) = \sqrt{\frac{G}{E}}\, dv(\mathbf{X}_u) = 0.$$

$$E_2(\theta_2) = \frac{1}{\sqrt{G}} \mathbf{X}_v(\sqrt{G}\, dv) = \frac{\sqrt{G}}{\sqrt{G}}\, dv(\mathbf{X}_v) = 1.$$

$$E_2(\theta_1) = \frac{1}{\sqrt{G}} \mathbf{X}_v(\sqrt{E}\, du) = \sqrt{\frac{E}{G}}\, du(\mathbf{X}_v) = 0. //$$

Calculate exterior derivatives towards finding f-la for ω_{12},

$$\begin{aligned} d\theta_1 &= d(\sqrt{E}\, du) \\ &= d\sqrt{E} \wedge du + \sqrt{E}\, d(du) \\ &= \left(\frac{\partial\sqrt{E}}{\partial u} du + \frac{\partial\sqrt{E}}{\partial v} dv\right) \wedge du \\ &= \cancel{(\sqrt{E})_v \frac{1}{\sqrt{E}}\, dv \wedge du} \\ &= \frac{-(\sqrt{E})_v}{\sqrt{G}}\, du \wedge \theta_2 \end{aligned}$$

$$d\theta_2 = d(\sqrt{G}\, dv) = (\sqrt{G})_{,u}\, du \wedge dv = \frac{-(\sqrt{G})_{,u}}{\sqrt{E}}\, dv \wedge \theta_1$$

However, 1st strud. eq$^{\text{ns}}$ state
$$d\theta_1 = \omega_{12} \wedge \theta_2$$
$$d\theta_2 = \omega_{21} \wedge \theta_1$$

Note, $\omega_{12} = \omega_{12}(E_1)\theta_1 + \omega_{12}(E_2)\theta_2$

and $d\theta_1 = \omega_{12} \wedge \theta_2 = \omega_{12}(E_1)\theta_1 \wedge \theta_2$

whereas $d\theta_2 = -\omega_{12}(E_2)\theta_2 \wedge \theta_1 = \cancel{\omega_{12}(E_2)\theta_1}$

Thus,

$$\omega_{12} = \omega_{12}(E_1)\theta_1 + \omega_{12}(E_2)\theta_2$$
$$= \left(+ \frac{(\sqrt{G})_{,u}}{\sqrt{E}} dv \right) - \left(\frac{(\sqrt{E})_{,v}}{\sqrt{G}} \right) du$$

By Lemma 5.1, the above must be ω_{12} as it is found by solving 1st structural eq^{n}s.

$$\boxed{\omega_{12} = \frac{-1}{\sqrt{G}} \frac{\partial}{\partial v}(\sqrt{E})\, du + \frac{1}{\sqrt{E}} \frac{\partial}{\partial u}(\sqrt{G})\, dv}$$

Example 6.2 p. 296 | (NOTE: this is way easier than shape operator methods)

Geographical coord. on sphere

$$E = r^2 \cos^2 v, \quad F = 0, \quad G = r^2$$

from $X(u,v) = (r\cos v \cos u,\ r\cos v \sin u,\ r\sin v)$ this is calculated by $E = X_u \cdot X_u$ and $G = X_v \cdot X_v$.

Therefore,

$$\theta_1 = \sqrt{E}\, du = r\cos(v)\, du$$
$$\theta_2 = \sqrt{G}\, dv = r\, dv$$

Clearly, $\frac{\partial}{\partial v}(\sqrt{E}) = \frac{\partial}{\partial v}(r\cos v) = -r\sin v$

whereas $\frac{\partial}{\partial u}(\sqrt{G}) = \frac{\partial}{\partial u}(r) = 0$ hence,

$$\omega_{12} = \frac{-1}{r}(-r\sin v)\, du = \sin(v)\, du$$

Thus $\boxed{\omega_{12} = \sin(v)\, du}$

Gaussian Curvature from intrinsic calculus

$$\omega_{12} = \frac{-(\sqrt{E})_v}{\sqrt{G}}du + \frac{(\sqrt{G})_u}{\sqrt{E}}dv$$

Recall, $d\omega_{12} = -K\,\theta_1\wedge\theta_2$ and we know $\theta_1 = \sqrt{E}\,du$ and $\theta_2 = \sqrt{G}\,dv$ hence $\theta_1\wedge\theta_2 = \sqrt{EG}\,du\wedge dv$. It remains to exterior differentiate ω_{12},

$$\begin{aligned} d\omega_{12} &= -\frac{\partial}{\partial v}\left[\frac{(\sqrt{E})_v}{\sqrt{G}}\right]dv\wedge du + \frac{\partial}{\partial u}\left[\frac{(\sqrt{G})_u}{\sqrt{E}}\right]du\wedge dv \\ &= \left(\left[\frac{(\sqrt{G})_u}{\sqrt{E}}\right]_u + \left[\frac{(\sqrt{E})_v}{\sqrt{G}}\right]_v\right)du\wedge dv \\ &= \frac{1}{\sqrt{EG}}\left(\left[\frac{(\sqrt{G})_u}{\sqrt{E}}\right]_u + \left[\frac{(\sqrt{E})_v}{\sqrt{G}}\right]_v\right)\theta_1\wedge\theta_2 \end{aligned}$$

The proposition below follows from $d\omega_{12} = -K\,\theta_1\wedge\theta_2$,

PROPOSITION 6.3: If $X : D \to M$ is orthogonal patch with $E = X_u\cdot X_u$ and $G = X_v\cdot X_v$ then the GAUSSIAN CURVATURE K is given by:

$$\boxed{K = \frac{-1}{\sqrt{EG}}\left(\left[\frac{(\sqrt{G})_u}{\sqrt{E}}\right]_u + \left[\frac{(\sqrt{E})_v}{\sqrt{G}}\right]_v\right)}$$

Again the isometric invariance of K is made manifest (see Lemma 4.5 where $F : M \to \bar{M}$ is shown to give $E = \bar{E}$, $F = \bar{F}$, $G = \bar{G}$ for isometry F)

§6.7 INTEGRATION & ORIENTATION

Def^{n}/ (7.1 p. 298) The interior R° of rectangle R: $a \le u \le b$, $c \le v \le d$ is the open set $a < u < b$, $c < v < d$. A 2-segment $\vec{X} : R \longrightarrow M$ is patchlike provided $\vec{X}|_{R^{\circ}} : R^{\circ} \longrightarrow M$ is a patch in M.

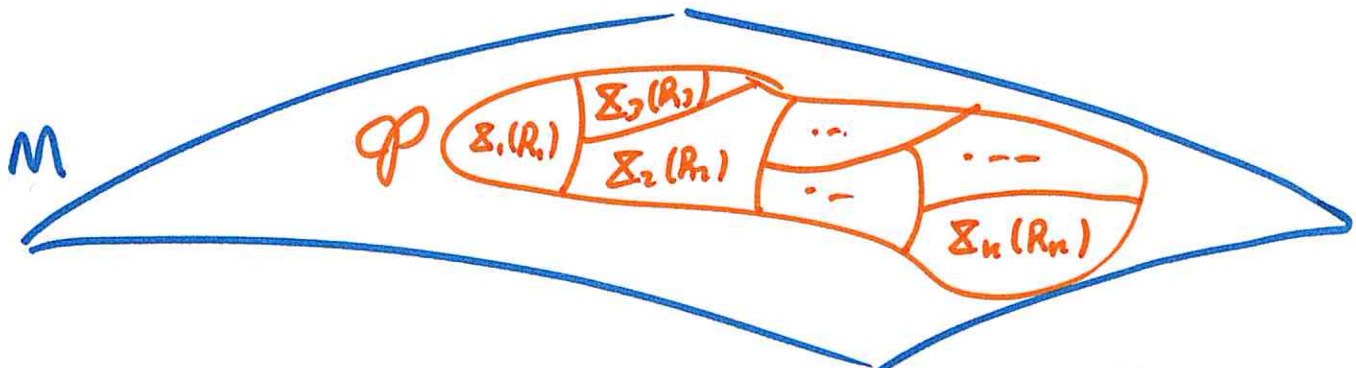
To calculate area we integrate $\sqrt{EG - F^2}$ over M.

$$\text{AREA} = \iint_D \sqrt{EG - F^2}\, du\, dv$$

Could be improper as D is open... so instead integrate over 2-segments (slight mod. of patch)

Example:

Def^n/ A paving of a region $\mathscr{P}$ in a surface M consists of finitely many patch-like 2-segments $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ whose images fill $\mathscr{P}$ such that $\bigcup_{i=1}^{n} \mathbf{x}_i(R_i) = \mathscr{P}$ and $p \in \mathscr{P} \Rightarrow p \in \mathbf{x}_i(R_i^\circ)$ for just one $i \in \mathbb{N}_n$.



Remark: an entire compact surface is always pavable.

Def^n/ An area form on a surface M is a differentiable 2-form μ whose value on any pair $V, W \in T_pM$

$$\mu(V,W) = \pm\sqrt{(V\cdot V)(W\cdot W) - (V\cdot W)^2} = \pm\|V\times W\|$$

The $\pm$ is not avoidable. $\exists$ two orientations locally for M.

Remark: $\mu(E_1, E_2) = \pm 1$ $\forall$ frames $\{E_1, E_2\}$ on M also suffices.

Lemma (7.5)(p. 301): A surface M has an area form iff it is orientable. On a connected surface there are <u>exactly</u> two area forms. We denote these by $\pm\, dM$.

This is nearly the definition of <u>orientable</u>. Recall M orientable iff $\exists\, \eta$ - 2-form on M where $\eta|_p \neq 0$ $\forall p \in M$. See Prop. 7.5 in Chpt. 4 for the rest, let's see, (p. 186 - 187 if time permits)

Prop. 7.5 says M orientable $\Longleftrightarrow$ $\exists\, U$ a unit normal vect. field on M

The fundamental identity there is:

$$\mu(v,w) = U \cdot (v \times w)$$

In this way U induces 2-form on M.

Def$^{\underline{n}}$/ If $\mathbf{x}$ is patchlike 2-segment in surface oriented by dM then,

$$\iint_{\mathbf{x}} dM = \iint_R dM(\mathbf{x}_u, \mathbf{x}_v)\, du\, dv$$

$(+)$ $\mathbf{x}$ is positively oriented (gives area of $\mathbf{x}(R)$)

$(-)$ $\mathbf{x}$ is negatively oriented. (gives $-$ area of $\mathbf{x}(R)$)

Area of paving $\mathcal{P}$ given by positively oriented paving

$$\text{AREA}(\mathcal{P}) = \sum_{i=1}^{k} \iint_{\mathbf{x}_i} dM .$$

Def^{n} (7.6) Let ν be a 2-form on a pavable oriented region $\mathcal{P} \subset M$ then the integral of ν over $\mathcal{P}$

$$\iint_{\mathcal{P}} \nu = \sum_{i=1}^{k} \iint_{X_i} \nu$$

where $X_1, X_2, \dots, X_k$ is positively oriented paving of $\mathcal{P}$.

To integrate $f : M \longrightarrow \mathbb{R}$ on $\mathcal{P}$ simply calculate

$$\iint_{\mathcal{P}} f \, dM$$

This we soon use to calculate $\iint K \, dM$ in next §.

Def^{n}/(8.1)(p. 304) Let K be the Gaussian curvature of a compact surface M oriented by area form dM.

$$\iint_M K \, dM = \text{total Gaussian curvature of } M.$$

We can calculate the total curvature of any pavable region $\mathcal{P}$ in the same way. Thus to calculate total curvature of M we

1.) find paving of M
2.) find total curvature for each 2-segment (which is assumed patchlike)

Then,

$$\iint_X K \, dM = \iint_R X^*(K \, dM) = \iint_R K(X) \, X^*(dM)$$

$$= \underbrace{\int_a^b \int_c^d}_{} K(X) \sqrt{EG - F^2} \, du \, dv$$

(typo here) → these should swap if $a \le u \le b$.

Example (8.2)(p. 305)

(1.) constant curvature:

$$K_{TOTAL} = \iint_M K\, dM = K \iint_M dM = K\ \text{AREA}(M)$$

$$K_{TOTAL}(\text{sphere}) = \frac{1}{r^2}(4\pi r^2) = \boxed{4\pi.}$$

$$K_{TOTAL}(\text{bugle}) = \frac{-1}{c^2}(2\pi c^2) = \boxed{-2\pi}$$

(2.) TORUS: Let $\mathbb{X}$ be 2-segment which covers T

$$\mathbb{X}^*(dT) = \sqrt{EG - F^2}\, du\, dv$$
$$= r(R + r\cos(u))\, du\, dv$$

However, we can calculate,

$$K(\mathbb{X}) = \frac{\cos(u)}{r(R + r\cos u)}$$

$$\therefore \iint_T K\, dT = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cos(u)\, du\, dv = \boxed{0.}$$

Apparently $K > 0$, $K < 0$ balance out on TORUS..

(3.) CATENOID: (based on Ex 7.1 of Chpt. 5)(p. 254-255)

$$\iint_{\mathbb{X}} K\, dM = -\int_0^a \int_0^{2\pi} \frac{du\, dv}{\cosh^2(u/c)} = -4\pi \tanh\left(\frac{a}{c}\right)$$

As $a \longrightarrow \infty$ find $K_{TOTAL} \longrightarrow \boxed{-4\pi}$

All integer multiples of 2π

CURIOUS...

Def$^{\underline{n}}$/(8.3) (p. 306) Let M, N be surfaces oriented by area forms dM and dN. Then the <u>Jacobian</u> of $F : M \longrightarrow N$ is the $\mathbb{R}$-valued function J_F on M such that

$$F^*(dN) = J_F \, dM$$

Let's calculate,

$$J_F(p)\, dM(v,w) = F^*(dN)(v,w) \xrightarrow{\text{def}^{\underline{n}} \text{ of pull-back of two-form.}} = dN(F^*(v), F^*(w))$$

Note, F regular iff $J_F(p) \neq 0 \quad \forall p \in M$.

F is orientation preserving at p if $J_F(p) > 0$.

F is orientation reversing at p if $J_F(p) < 0$.

Moreover,

$$\underbrace{|J_F(p)|}_{\text{rate at which } F \text{ is expanding area at } P.} \, |dM(v,w)| = |dN(F_*(v), F_*(w))|$$

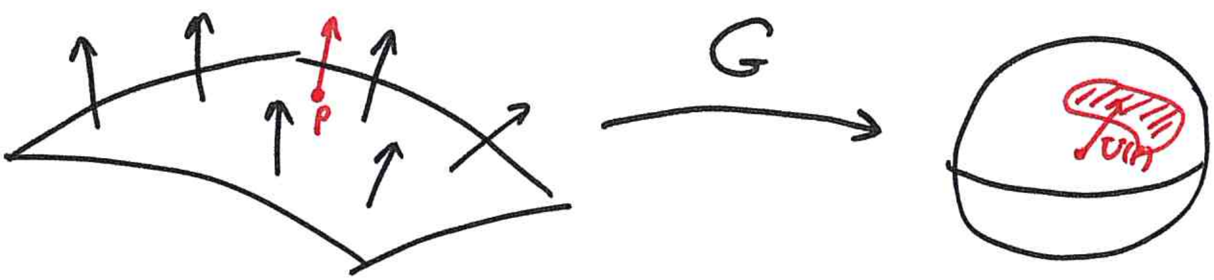
$$\iint_M J_F \, dm = \iint_M F^*(dN) = \text{signed area of } F(M) \text{ a.k.a. algebraic area.}$$

Can be (+), (−) depending on orientation of F ($J_F < 0$ vs. $J_F > 0$)

Gauss Map

$$G : M \longrightarrow \Sigma = \text{unit-sphere.}$$

$G(p) = U(p) \leftarrow$ unit-normal to M at p.



Th^m (8.4) THE GAUSSIAN CURVATURE, K OF AN ORIENTED SURFACE $M \subset \mathbb{R}^3$ IS THE JACOBIAN OF ITS GAUSS MAP

Proof: If $U = \sum g_i U_i$ then $G = (g_1, g_2, g_3)$.
Recall the SHAPE OPERATOR,

$$-S(v) = \nabla_v U = \sum v[g_i] U_i$$

Thus, by Prop. 7.5 on pg. $F_*(v) = (v[f_1], .., v[f_n])_{F(p)}$.

$$G_*(v) = \sum_{i=1}^{3} v[g_i] U_i(G(p)) \quad (\star)$$

Thus $-S(v) \parallel G_*(v)$. We seek to show $K dM = G^*(d\Sigma)$

$$\begin{aligned}(K dM)(v,w) &= K(p)\, dM(v,w) \\ &= K(p)\, U(p) \cdot (v \times w) \\ &= U(p) \cdot S(v) \times S(w)\end{aligned}$$

Lemma 3.4 of Chpt. 5

Likewise,

$$\begin{aligned}G_*(d\Sigma)(v,w) &= d\Sigma(G_* v, G_* w) \\ &= \bar{U}(G(p)) \cdot G_*(v) \times G_*(w) \\ &= U(p) \cdot S(v) \times S(w) \quad \text{by } \star. \, /\!/\end{aligned}$$