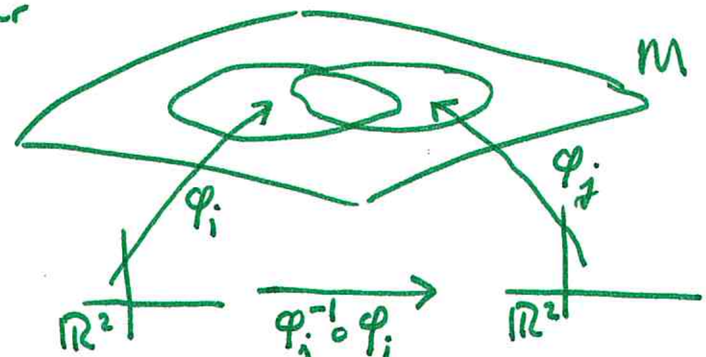
CHAPTER 7 : RIEMANNIAN GEOMETRY

Def^{n}/ M is an abstract surface if $\exists$ patches $\varphi_i : D_i \longrightarrow U_i \subset M$ s.t. $\bigcup_i U_i = M$ and φ_i, φ_j, $i \neq j$ are smoothly related (compatible) whenever $U_i \cap U_j \neq \phi$; in particular



$\underbrace{\varphi_j^{-1} \circ \varphi_i}$ must be smooth mapping on $\mathbb{R}^2$.

We say M is a geometric surface if M is paired with a metric $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ at each $p \in M$ and $p \longmapsto g_p$ is smooth. The metric $g_p(v,w) = \langle v, w \rangle$ (a usual notation) must be an inner product on T_pM,

1.) $\langle v, w \rangle = \langle w, v \rangle$

2.) $\langle cv_1 + v_2, w \rangle = c\langle v_1, w \rangle + \langle v_2, w \rangle$
$\langle w, cv_1 + v_2 \rangle = c\langle w, v_1 \rangle + \langle w, v_2 \rangle$

3.) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$.

These requirements give us Riemannian geometry. If 1.) or 3.) is modified other types of geometry (symplectic, or Lorentzian etc...) may be obtained.

Remark: replace $\mathbb{R}^2$ with $\mathbb{R}^n$ you get an n-dim'l Riemannian manifold modulo some topological comments.

Given a metric $\langle , \rangle$ we may define lengths and angles ~~etc~~ as follows:

Def$^{\underline{n}}$/ $||v|| = \sqrt{\langle v,v \rangle}$ & $\underbrace{\cos\theta = \frac{\langle v,w \rangle}{||v|| \, ||w||}}_{\text{for } v,w \neq 0.}$

Methods to Construct Metrics

(1.) Distort old metric to form new. On $\mathbb{R}^2$,

$$\langle v, w \rangle = \frac{v \cdot w}{h^2(p)}$$

For $h > 0$ defines geometric surface M <u>conformal</u> <u>with</u> <u>ruler function h</u>

<u>Claim</u>: "<u>locally</u>" every geometric surface is so expressed.

(2.) <u>Pullback</u>: If M is an abstract surface and N is a geometric surface with metric g then we define, via $F: M \to N$ a <u>regular</u> map

$$\langle v, w \rangle_M = \langle F_*(v), F_*(w) \rangle_N$$
$$= g(dF(v), dF(w)).$$

Bilinearity of $\langle , \rangle_M$ is clear from linearity of dF and bilinearity of g. Symmetry & positive definite props also transfer directly from g to $\langle , \rangle_M$.

<u>Notation</u>: $\langle , \rangle_M = F^*(g)$.

<u>NOTE</u>: $\langle v, w \rangle_M = \langle F_*(v), F_*(w) \rangle$ hence $(M, F^*(g))$ is <u>isometric</u> to (N, g). We should refer to a geometric surface as a pair since we'll soon see the same point set supports multiple geometries! The metric determines the geometry.

(3.) Given functions E, F, G on abstract surface M with suitable properties ($E, G, EG > F^2$) we can construct $\langle , \rangle$ by imposing

$$\langle X_u, X_u \rangle = E$$
$$\langle X_u, X_v \rangle = F$$
$$\langle X_v, X_v \rangle = G$$

See Exercise 4. For some indication on <u>how</u> to do this. (I'm a bit stuck on those details at moment.)

We now work on describing how our intrinsic calculus on $M \subset \mathbb{R}^3$ is naturally carried to defining geometric objects (frames, connection form, curvature ...) on <u>M</u> abstract.

Def$^{\underline{n}}$/ Frame field on $(M, \langle , \rangle)$ consists of two <u>orthonormal</u> vector fields E_1, E_2 defined on some open subset of M. Here we should say $\langle , \rangle$-orthonormal to emphasize the $\langle , \rangle$-dependence:

$$\langle E_1, E_1 \rangle = 1, \quad \langle E_2, E_2 \rangle = 1, \quad \langle E_1, E_2 \rangle = 0.$$

Dual frames also determined same as before:

Def$^{\underline{n}}$/ Given frame field E_1, E_2 on $(M, \langle , \rangle)$ we say θ_1, θ_2 is dual frame to E_1, E_2 iff $\theta_i(E_j) = \delta_{ij}$ and $\theta_1^{(p)}, \theta_2^{(p)} : T_pM \to \mathbb{R}$ linear for each $p \in$ ~~M~~ $\mathcal{U}$ where $\mathcal{U}$ is the open set on which E_1, E_2 is frame.

Def^{n}/ Given frame E_1, E_2 with coframe θ_1, θ_2 on $(M, \langle\, , \rangle)$ we define ω_{12} implicitly via

$$d\theta_1 = \omega_{12} \wedge \theta_2 \quad \& \quad d\theta_2 = \omega_{21} \wedge \theta_1$$

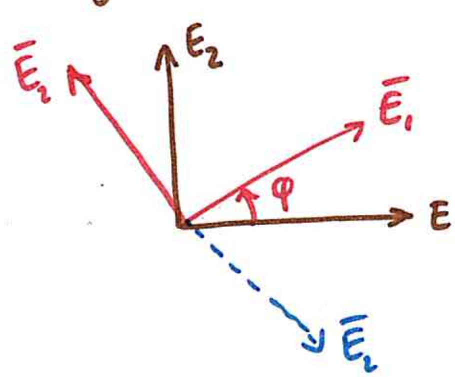
To be explicit, $\omega_{12} = \omega_{12}(E_1)\,\theta_1 + \omega_{12}(E_2)\,\theta_2$

and $\omega_{12} \wedge \theta_1 = \omega_{12}(E_2)\,\theta_2 \wedge \theta_1 = -d\theta_2 \quad \therefore \; d\theta_2 = \omega_{12}(E_2)\,\theta_1 \wedge \theta_2$

$\omega_{12} \wedge \theta_2 = \omega_{12}(E_1)\,\theta_1 \wedge \theta_2 = d\theta_1$

Hence, $\underline{\omega_{12} = \big(d\theta_1(E_1, E_2)\big)\theta_1 + \big(d\theta_2(E_1, E_2)\big)\theta_2}$.

We need to investigate the coordinate dependence of $\theta_1, \theta_2, \omega_{12}$ ... if we had $\bar{E}_1, \bar{E}_2$ another frame then how will $\bar{\theta}_1, \bar{\theta}_2, \bar{\omega}_{12}$ relate to the original coord. system's forms?



← both frames orthonormal $\Rightarrow \exists\, \varphi$ s.t.

$$\bar{E}_1 = \cos\varphi\, E_1 + \sin\varphi\, E_2$$

$$\left.\begin{aligned} \bar{E}_2 &= -\sin\varphi\, E_1 + \cos\varphi\, E_2 \\ \bar{E}_2{}' &= \sin\varphi\, E_1 - \cos\varphi\, E_2 \end{aligned}\right\} \text{either choice gives}$$

$\langle \bar{E}_i, \bar{E}_j \rangle = \delta_{ij}$

Note, $\bar{E}_1, \bar{E}_2$ have same orientation as E_1, E_2

However, $\bar{E}_1, \bar{E}_2{}'$ have opposite orientation as E_1, E_2

LEMMA (1.4): Let E_1, E_2 and $\bar{E}_1, \bar{E}_2$ be frame fields on the same region in M. If these frame fields have:

(1.) the same orientation, then

$$\bar{\omega}_{12} = \omega_{12} + d\varphi \quad \text{and} \quad \bar{\theta}_1 \wedge \bar{\theta}_2 = \theta_1 \wedge \theta_2$$

(2.) opposite orientation, then

$$\bar{\omega}_{12} = -(\omega_{12} + d\varphi) \quad \text{and} \quad \bar{\theta}_1 \wedge \bar{\theta}_2 = -\theta_1 \wedge \theta_2$$

Proof: (1.)

$$\bar{E}_1 = \cos\varphi\, E_1 + \sin\varphi\, E_2$$
$$\bar{E}_2 = -\sin\varphi\, E_1 + \cos\varphi\, E_2$$

$$\theta_1 = \theta_1(\bar{E}_1)\bar{\theta}_1 + \theta_1(\bar{E}_2)\bar{\theta}_2 = \cos\varphi\, \bar{\theta}_1 - \sin\varphi\, \bar{\theta}_2$$
$$\theta_2 = \theta_2(\bar{E}_1)\bar{\theta}_1 + \theta_2(\bar{E}_2)\bar{\theta}_2 = \sin\varphi\, \bar{\theta}_1 + \cos\varphi\, \bar{\theta}_2$$

Observe,

$$\begin{aligned} \theta_1 \wedge \theta_2 &= (\cos\varphi\, \bar{\theta}_1 - \sin\varphi\, \bar{\theta}_2) \wedge (\sin\varphi\, \bar{\theta}_1 + \cos\varphi\, \bar{\theta}_2) \\ &= \cos^2\theta\, \bar{\theta}_1 \wedge \bar{\theta}_2 - \sin^2\varphi\, \bar{\theta}_2 \wedge \bar{\theta}_1 \\ &= \bar{\theta}_1 \wedge \bar{\theta}_2. \end{aligned}$$

Also, ext. diff to obtain,

$$d\theta_1 = -\sin\varphi\, d\varphi \wedge \bar{\theta}_1 - \cos\varphi\, d\varphi \wedge \bar{\theta}_2 + \cos\varphi\, d\bar{\theta}_1 - \sin\varphi\, d\bar{\theta}_2$$
$$d\theta_2 = \cos\varphi\, d\varphi \wedge \bar{\theta}_1 - \sin\varphi\, d\varphi \wedge \bar{\theta}_2 + \sin\varphi\, d\bar{\theta}_1 + \cos\varphi\, d\bar{\theta}_2$$

Oh, now $d\bar{\theta}_1 = \bar{\omega}_{12} \wedge \bar{\theta}_2$ and $d\bar{\theta}_2 = \bar{\omega}_{21} \wedge \bar{\theta}_1$ hence,

$$\begin{aligned} d\theta_1 &= -\sin\varphi\, d\varphi \wedge \bar{\theta}_1 - \cos\varphi\, d\varphi \wedge \bar{\theta}_2 \\ &\quad + \sin\varphi\, \bar{\omega}_{12} \wedge \bar{\theta}_2 + \cos\varphi\, \bar{\omega}_{21} \wedge \bar{\theta}_1 \\ &= (\cos\varphi\, \bar{\omega}_{21} - \sin\varphi\, d\varphi) \wedge \bar{\theta}_1 + (\sin\varphi\, \bar{\omega}_{12} - \cos\varphi\, d\varphi) \wedge \bar{\theta}_2 \\ &= \end{aligned}$$

Proof of $\bar{\omega}_{12} = \omega_{12} + d\varphi$:

Recall $\theta_1 = \cos\varphi\,\bar{\theta}_1 - \sin\varphi\,\bar{\theta}_2$ & $\theta_2 = \sin\varphi\,\bar{\theta}_1 + \cos\varphi\,\bar{\theta}_2$

Thus,

$$
\begin{aligned}
d\theta_1 &= (-\sin\varphi\,\bar{\theta}_1 - \cos\varphi\,\bar{\theta}_2)\wedge(-d\varphi) + \cos\varphi\,d\bar{\theta}_1 - \sin\varphi\,d\bar{\theta}_2 \\
&= (\sin\varphi\,\bar{\theta}_1 + \cos\varphi\,\bar{\theta}_2)\wedge d\varphi + \cos\varphi\,d\bar{\theta}_1 - \sin\varphi\,d\bar{\theta}_2 \\
&= \theta_2\wedge d\varphi + \cos\varphi\,\bar{\omega}_{12}\wedge\bar{\theta}_2 - \sin\varphi\,\bar{\omega}_{21}\wedge\bar{\theta}_1 \\
&= \theta_2\wedge d\varphi - \cos\varphi\,\bar{\theta}_2\wedge\bar{\omega}_{12} - \sin\varphi\,\bar{\theta}_1\wedge\bar{\omega}_{12} \\
&= \theta_2\wedge d\varphi - (\cos\varphi\,\bar{\theta}_2 + \sin\varphi\,\bar{\theta}_1)\wedge\bar{\omega}_{12} \\
&= \theta_2\wedge d\varphi - \theta_2\wedge\bar{\omega}_{12} \\
&= (\bar{\omega}_{12} - d\varphi)\wedge\theta_2
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
d\theta_2 &= d\varphi\wedge(\cos\varphi\,\bar{\theta}_1 - \sin\varphi\,\bar{\theta}_2) + \sin\varphi\,d\bar{\theta}_1 + \cos\varphi\,d\bar{\theta}_2 \\
&= d\varphi\wedge\theta_1 + \sin\varphi\,\bar{\omega}_{12}\wedge\bar{\theta}_2 + \cos\varphi\,\bar{\omega}_{21}\wedge\bar{\theta}_1 \\
&= d\varphi\wedge\theta_1 + \bar{\omega}_{12}\wedge(\underbrace{\sin\varphi\,\bar{\theta}_2 - \cos\varphi\,\bar{\theta}_1}_{(-)}) \\
&= d\varphi\wedge\theta_1 - \bar{\omega}_{12}\wedge\theta_1 \\
&= (\bar{\omega}_{12} - d\varphi)\wedge\theta_1
\end{aligned}
$$

Hence $\omega_{12} = \bar{\omega}_{12} - d\varphi$ // (Recall from Chpt. 6, satisfying ~~the~~ 1st structural eq$^{\underline{s}}$ uniquely determines ω_{12})

Remark: $\omega_{12} = \bar{\omega}_{12} - d\varphi$ shows the choice of frame is tied to choice of angle...

In the abstract case we have no shape operator so $\det(S) = K$ is not going to work. However, the 2^{nd} structural eq^{n} generalizes nicely,

> **Th^{m} (2.1)** Given $(M, \langle , \rangle)$ there is a unique real-valued function K s.t. for every frame field on M the 2^{nd} structural eq^{n}
> $$d\omega_{12} = -K\,\theta_1 \wedge \theta_2$$
> holds. We call K the GAUSSIAN CURVATURE OF M

Proof: For a given frame field, two-dim'l calculus reveals $\exists K$ s.t. $d\omega_{12} = -K\,\theta_1 \wedge \theta_2$ (ω_{12} is 1-form so this is inevitable as $\theta_1 \wedge \theta_2$ is the only two-form.) Likewise, for $\bar{E}_1, \bar{E}_2$ frame, $d\bar{\omega}_{12} = -\bar{K}\,\bar{\theta}_1 \wedge \bar{\theta}_2$. We need to show $K = \bar{K}$ if the frames domains overlap. Recall from last section $\bar{\omega}_{12} = \pm(\omega_{12} + d\varphi)$ and $\bar{\theta}_1 \wedge \bar{\theta}_2 = \pm(\theta_1 \wedge \theta_2)$

+ (oriented same way)
− (reverse oriented)

Consider then,

$$\begin{aligned}
d\bar{\omega}_{12} &= d[\pm(\omega_{12} + d\varphi)] \\
&= \pm\, d\omega_{12} + d(d\varphi) \\
&= \mp K\,\theta_1 \wedge \theta_2 \\
&= \mp K(\pm\,\bar{\theta}_1 \wedge \bar{\theta}_2) \\
&= -K\,\bar{\theta}_1 \wedge \bar{\theta}_2 = -\bar{K}\,\bar{\theta}_1 \wedge \bar{\theta}_2 \Rightarrow \underline{K = \bar{K}}.//
\end{aligned}$$

Remark: K is determined w/o regard to the orientation of the frame. We can calculate K for non-orientable surfaces.

Frame Geometry Eq's

$$d\theta_1 = \omega_{12} \wedge \theta_2$$
$$d\theta_2 = \omega_{21} \wedge \theta_1$$
$$d\omega_{12} = -K\,\theta_1 \wedge \theta_2$$

TRIVIAL EXAMPLE:

$M = \mathbb{R}^2$ with U_1, U_2 frame has coframe dx, dy
$\theta_1 = dx$ and $\theta_2 = dy$. Clearly $d\theta_1 = 0$, $d\theta_2 = 0$
thus $\omega_{12} = 0$ and $d\omega_{12} = 0 \Rightarrow \underline{K = 0}$.

EXAMPLE (2.2) A FLAT TORUS $(\underbrace{X(u,v) = ((R+r\cos u)\cos v,\ (R+r\cos u)\sin v,\ r\sin u)}$,

Let T be torus of revolution and define

$$\langle X_u, X_u \rangle = 1$$
$$\langle X_u, X_v \rangle = 0$$
$$\langle X_v, X_v \rangle = 1$$

(fix $u = u_o$ get circle in $z = r\sin u_o$ plane of radius $R + r\cos u_o$.)

these define $(T, \langle , \rangle)$ or simply T_o. Observe
$X_*(U_1) = X_u$ and $X_*(U_2) = X_v$ hence
$K(T_o) = K(\mathbb{R}^2) = 0$, it's isometric to plane,
I should clarify the Euclidean plane.

Remark: $T_o \subset \mathbb{R}^3$ however, $\langle , \rangle$ is
not induced from Euclidean metric on $\mathbb{R}^3$.

Remark: T_0 is compact subset of $\mathbb{R}^3$ as constructed thus (if given geometry induced from its shape in $\mathbb{R}^3$ meaning the metric of T_0 were induced from $\mathbb{R}^3$) cannot be everywhere flat ($K = 0$ on T_0) as Thm (3.5) of Chpt. 6 $\Rightarrow$ $K(P) > 0$ at at least one point of T_0 $\therefore$ the metric we gave to T_0 cannot be realized as a restriction of the Euclidean metric on $\mathbb{R}^3$.

BIG PICTURE:

- BEFORE CHAPTER 7: STUDIED SURFACES IN $\mathbb{R}^3$ WHOSE METRIC WAS INDUCED FROM METRIC ON $\mathbb{R}^3$
- CHAPTER 7 and Beyond: METRIC GIVEN TO ABSTRACT SURFACE NEED NOT BE INDUCED FROM AMBIENT CONTEXT. However, it can be as all surfaces studied in previous Chapters are also GEOMETRIC SURFACES

Remark: Given a geometric surface M when does $\exists n \in \mathbb{N}$ and $\bar{M} \subset \mathbb{R}^n$ for which $F: M \to \bar{M} \subseteq \mathbb{R}^n$ is an isometry? Just because $n = 3$ fails for torus ~~embedding~~ doesn't mean it cannot fit inside larger $\mathbb{R}^n$... see Whitney embedding Thm; and I think for our context, NASH gave the isometric version (Whitney just focused on abstract surface, no geometry)

Cor (2.3): If $\mathbb{R}^2$ has $\langle v, w\rangle_p = \frac{1}{h^2(p)}(v \cdot w)$ then,

$$K = h(h_{uu} + h_{vv}) - (h_u^2 + h_v^2) = h^2 \Delta \log h$$

<u>Proof</u>: Note $E = 1/h^2 = G$ and $F = 0$. Also we found for orthogonal patches ($F=0$) that

$$K = \frac{-1}{\sqrt{EG}}\left(\frac{\partial}{\partial u}\left[\frac{1}{\sqrt{E}}\frac{\partial}{\partial u}\left[\sqrt{G}\right]\right] + \frac{\partial}{\partial v}\left[\frac{1}{\sqrt{G}}\frac{\partial}{\partial v}\left[\sqrt{E}\right]\right]\right)$$

$$= -h^2\left(\frac{\partial}{\partial u}\left[h\left[\frac{-1}{h^2}\frac{\partial h}{\partial u}\right]\right] + \frac{\partial}{\partial v}\left[h\frac{\partial}{\partial v}\left[\frac{1}{h}\right]\right]\right)$$

$$= h^2\left(\underline{\frac{\partial}{\partial u}\left[\frac{1}{h}\frac{\partial h}{\partial u}\right] + \frac{\partial}{\partial v}\left[\frac{1}{h}\frac{\partial h}{\partial v}\right]}\right) *$$

$$= h^2\left[\frac{-1}{h^2}\left(\frac{\partial h}{\partial u}\right)^2 + \frac{1}{h}\frac{\partial^2 h}{\partial u^2} + \left(\frac{-1}{h^2}\right)\left(\frac{\partial h}{\partial v}\right)^2 + \frac{1}{h}\frac{\partial^2 h}{\partial v^2}\right]$$

$$= -\left(\frac{\partial h}{\partial u}\right)^2 - \left(\frac{\partial h}{\partial v}\right)^2 + h\left(\frac{\partial^2 h}{\partial u^2} + \frac{\partial^2 h}{\partial v^2}\right). //$$

The fact this can be expressed as $h^2 \Delta \log(h)$ follows from Ex. #2 of §6.6 where we learn $\Delta f = f_{uu} + f_{vv}$ the <u>LAPLACIAN</u>.

$$\Delta \log(h) = \frac{\partial^2}{\partial u^2}\left[\log(h)\right] + \frac{\partial^2}{\partial v^2}\left[\log(h)\right]$$

$$= \underline{\frac{\partial}{\partial u}\left[\frac{1}{h}\frac{\partial h}{\partial u}\right] + \frac{\partial}{\partial v}\left[\frac{1}{h}\frac{\partial h}{\partial v}\right]} *$$

Hence $\boxed{K = h^2 \Delta \log(h)}$. //

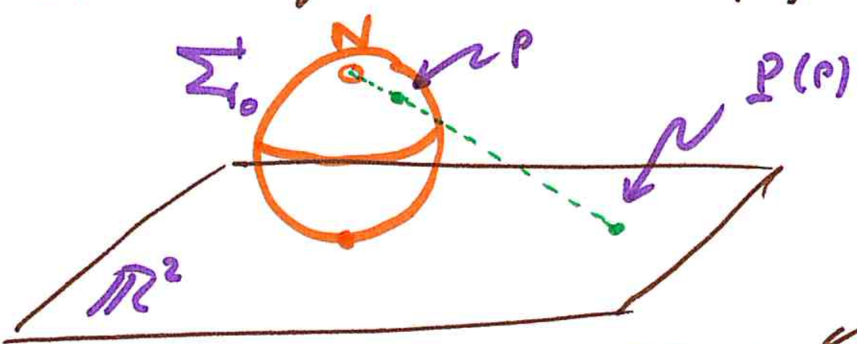
<u>Remark</u>: $\mathbb{R}^2$ with $X(u,v) = (u,v)$ and $\langle\ ,\ \rangle$ above is <u>conformal</u>, I mean the "patch is conformal" but isometric to Euclidean plane.
See Remark 1.3 pg. 323.

Example 2.4 (pg. 331)

(1.) THE STEREOGRAPHIC SPHERE (see Ex. 5.5 of Chpt. 4)

- Let Σ' be unit sphere resting on xy-plane with center at $(0,0,1)$. Let $\underbrace{\Sigma'_0 = \Sigma' - \{\overset{N}{(0,0,2)}\}}_{\text{punctured sphere, is missing North Pole}}$

- imagine light source at top N and for each $p \in \Sigma'$ let $P(p)$ be shadow of p in the xy-plane. Here we identify $\mathbb{R}^2$ with $\mathbb{R}^2 \times \{0\}$ for convenience.



$$P(p_1, p_2, p_3) = \left(\frac{2p_1}{2-p_3}, \frac{2p_2}{2-p_3} \right)$$

page 167 has derivation. It's similar Δ and some trig/algebra.

Clearly $P : \Sigma'_0 \longrightarrow \mathbb{R}^2$ is smooth as $p_3 \neq 2$ for $p \in \Sigma'_0$. In fact $P^{-1} : \mathbb{R}^2 \longrightarrow \Sigma'_0$ is likewise smooth,

$$P^{-1}(x,y) = (p_1, p_2, p_3)$$

Just solve $x = \dfrac{2p_1}{2-p_3}$ and $y = \dfrac{2p_2}{2-p_3}$

for p_1, p_2, p_3 given $p_3 \geq 0$, $p_1^2 + p_2^2 + (p_3 - 1)^2 = 1$.

$$x(2-p_3) = 2p_1 \longrightarrow 2p_1 + xp_3 = 2x$$
$$y(2-p_3) = 2p_2 \longrightarrow 2p_2 + yp_3 = 2y$$

etc... it's just algebra ☺.

Remark: remind me to give handout from pages 120-122 and Chpt. 15... of John McCleary's "GEOMETRY FROM A DIFFERENTIAL VIEWPOINT"

(1.) STEREOGRAPHIC SPHERE CONTINUED:

- pull-back the Euclidean metric $\underbrace{ds^2 = dx^2 + dy^2}_{g_{\text{Euclidean}}}$ on $\mathbb{R}^2$ under the steographic projection map $P : \Sigma_0 \longrightarrow \mathbb{R}^2$. It follows the abstract surface Σ_0 paired with $P^*(g_{\text{Euclid.}})$ is FLAT ! $K(\Sigma_0) = 0$.
- "intrinsically" Σ_0 is as flat as the Euclidean plane.

(2.) The STEREOGRAPHIC PLANE: Now turn the tables and pull-back the usual induced (curved) metric on Σ_0 to the plane $\mathbb{R}^2$ (originally thought of w/o geometry!) to obtain $(\mathbb{R}^2, (P^{-1})^*(g_{\text{sphere}}))$ the

STEREOGRAPHIC PLANE.

- intrinsically this non-euclidean plane is curved just like the sphere Σ_0, it has constant Gaussian curvature $K = 1$.

CALCULATION: (see 332) you can show

$$\langle v, w \rangle = (P^{-1})_*(v) \cdot (P^{-1})_*(w) = \left(1 + \frac{\|q\|^2}{4}\right) v \cdot w$$

Hence the steographic plane is conformally related to the Eucidean plane with

$$h(u,v) = 1 + \frac{u^2 + v^2}{4}$$

Rulers get longer as they move further from origin. Small circle at N on Σ_0 gives stupidly large circle in steographic plane. See nbhds of ∞ in $\mathbb{C}$ - variables...

Example 2.5 (THE HYPERBOLIC PLANE)

Use ruler function $h = 1 - \frac{u^2+v^2}{4}$ as opposed to (+) for stereographic plane...

Since $h > 0$ is needed we consider $u^2+v^2 < 4$ given this non-euclidean geometry: the

(H) Poincaré disk model of hyperbolic plane

$$\langle v, w \rangle = \frac{v \cdot w}{h^2} \qquad \text{for } v, w \in \{(u,v) \mid u^2+v^2 < 4\}$$

(oneil does not give us the mapping from $\mathbb{R}^2 \longrightarrow (H, \langle , \rangle)$ which is an isometry) *

However, we showed $K = h^2 \Delta \log h$ hence

$$K = \xcancel{h^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\log\left(1 - \frac{u^2}{4} - \frac{v^2}{4}\right)}$$

why? instead

$$= h(h_{uu} + h_{vv}) - h_u^2 + h_v^2$$

$$= \left(1 - \tfrac{1}{4}(u^2+v^2)\right)\left(-\tfrac{1}{2} - \tfrac{1}{2}\right) + \frac{u^2}{4} + \frac{u^2}{4}$$

$$= \underline{-1}.$$

Despite * we've said something interesting