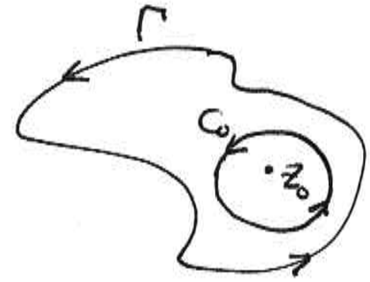
Computation of Laurent Series and $\int$-techniques

If z_0 is an isolated singularity then on some punctured disk about z_0 we have:

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-z_0)^j$$

Moreover, if Γ is a closed, positively oriented contour, which contains isolated singular pt. z_0 then



$$\oint_\Gamma f(z)\,dz = \oint_{C_0} f(z)\,dz$$

$$= \sum_{j=-\infty}^{\infty} \oint_{C_0} a_j (z-z_0)^j\,dz$$

$$= \sum_{j=-\infty}^{\infty} a_j \underbrace{\oint (z-z_0)^j\,dz}_{2\pi i\,\delta_{j,-1}}$$ ← Homework Problem.

$$= 2\pi i\, a_{-1}$$

We can calculate $\oint_\Gamma f(z)dz$ by picking-off the $j=-1$ coeff. of the Laurent series for $f(z)$.

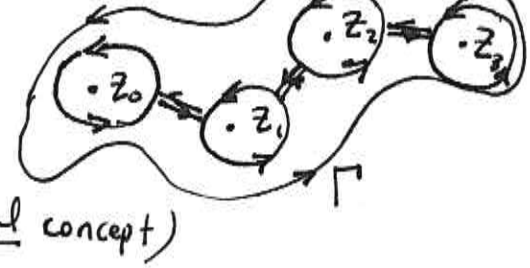
$$\boxed{\text{Def}^n/\ \ \text{Res}(f;\, z_0) = a_{-1}}$$

If we have multiple singularities $z_0, z_1, \dots, z_n$ then

$$\oint_\Gamma f(z)\,dz = \int_{C_0} f\,dz + \int_{C_1} f\,dz + \cdots + \int_{C_n} f\,dz$$ (I argued this on Test 1 essentially)

$$= 2\pi i\,\text{Res}(z_0) + 2\pi i\,\text{Res}(z_1) + \cdots 2\pi i\,\text{Res}(z_n)$$

$$= 2\pi i \sum_{j=1}^{n} \text{Res}(z_j)$$



Remark: we'll need a method to find an expansion around each singularity (Laurent exp. is <u>local</u> concept)

- I'll begin with a few simple examples which involve algebraic manipulation of known results.

$\boxed{E100}$ $f(z) = \frac{1}{z^4}\sin(z) = \frac{1}{z^4}\left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \cdots\right)$

Thus $f(z) = \frac{1}{z^3} - \frac{1}{6z} + \frac{1}{120}z^2 + \cdots$

It follows $\underline{\text{Res}(f;0) = -1/6}$.

$\boxed{E101}$ $f(z) = \frac{1}{z}\sin(z)$ find Laurent expansion at $z=1$

$$f(z) = \frac{1}{z-1+1}\sin(z-1+1)$$

$$= \left(\frac{1}{1-(1-z)}\right)\sin(1+(z-1))$$

$$= \sum_{n=0}^{\infty}(1-z)^n\left[\sin(1)\cos(z-1) + \cos(1)\sin(z-1)\right]$$

$$= \sum_{n=0}^{\infty}(-1)^n(z-1)^n\left[\sin(1)\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}(z-1)^{2j} + \cos(1)\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(z-1)^{2k+1}\right]$$

We could continue and multiply (gulp.) these series, but it's already clear that $\text{Res}(f;1) = 0$. We'll soon find a more efficient way to see this!

$\boxed{E102}$ $f(z) = z^5\sin\left(\frac{1}{z^2}\right)$

$$= z^5\left(\frac{1}{z^2} - \frac{1}{6}\left(\frac{1}{z^2}\right)^3 + \frac{1}{120}\left(\frac{1}{z^2}\right)^5 - \cdots\right)$$

$$= z^3 - \frac{1}{6}\left(\frac{1}{z}\right) + \frac{1}{120}\frac{1}{z^5} + \cdots$$

$\underline{\text{Res}(f(z);0) = -1/6}$.

Remark: I worked a few more of these including a nice three case I, II, III geometric series driven ex. Look those up.

$\boxed{\text{Def}^{\underline{n}}/ \text{ A } \underline{\text{simple}}\ \underline{\text{pole}} \text{ is an isolated singularity of } f(z) \text{ for which } f(z) = \sum_{j=-1}^{\infty} a_j (z-z_0)^j.}$

$$f(z) = \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

$$(z-z_0)f(z) = a_{-1} + a_0(z-z_0) + a_1(z-z_0)^2 + \cdots$$

thus, for a simple pole:

$$\boxed{\text{Res}(f; z_0) = a_{-1} = \lim_{z \to z_0} (z-z_0)f(z)}$$

$\boxed{E103}$ $f(z) = \dfrac{\sin(z)}{z^2}$ at $z_0 = 0$ we have simple pole

$$\text{Res}(f; 0) = \lim_{z \to 0}\left[z \frac{\sin z}{z^2}\right] = \lim_{z \to 0}\left[\frac{\sin z}{z}\right] = 1.$$

$\boxed{E104}$ $f(z) = \dfrac{e^z}{(z-1)(z-2)}$ has simple poles $z=1$, $z=2$.

$$\text{Res}(f; 1) = \lim_{z \to 1}(z-1)f(z) = \lim_{z \to 1}\left(\frac{e^z}{z-2}\right) = \frac{e}{-1} = \boxed{-e.}$$

$$\text{Res}(f; 2) = \lim_{z \to 2}(z-2)f(z) = \lim_{z \to 2}\left(\frac{e^z}{z-1}\right) = \frac{e^2}{1} = e^2$$

- These examples beg the question how do I know the given pt. is in fact a simple pole? Essentially the observation is that the numerator is analytic whereas the denominator had a simple zero at the point.

$\boxed{E105}$ Suppose $f(z) = \dfrac{P(z)}{Q(z)}$ where $P(z_0) \neq 0$, $Q(z_0) = 0$ and P, Q are analytic at z_0, it follows z_0 is simple pole ∴

$$\therefore \ \text{Res}(f(z); z_0) = \lim_{z \to z_0}(z-z_0)\frac{P(z)}{Q(z)} = \lim_{z \to z_0} \frac{P(z)}{\frac{Q(z)-Q(z_0)}{z-z_0}} = \frac{P(z_0)}{Q'(z_0)}$$

nice f-la!

From Example 105 we find:

See §57 of Churchill (pg. 195-196)

$$\boxed{\text{Res}\left(\frac{P(z)}{Q(z)}\,;\,z_0\right) = \frac{P(z_0)}{Q'(z_0)}} \leftarrow \underline{PQ\text{-f'la}}$$

provided $P(z_0) \neq 0$ and $Q(z_0) = 0$ where $Q(z)$ has a simple zero at z_0.

<u>Remark</u>: $\boxed{E103}$ & $\boxed{E104}$ can be calculated by the <u>PQ-f-la.</u>

$\boxed{E106}$ $f(z) = \tan(2z)$, find $\text{Res}(f(z)\,;\,z_0)$ for each singularity z_0 of $f(z) = \tan(2z) = \dfrac{\sin(2z)}{\cos(2z)}$.

Identify $\cos(2z) = Q(z)$ & $\sin(2z) = P(z)$. Moreover,

$\cos\Theta = 0$ iff $\Theta = \frac{\pi}{2}(2n+1)$, $n \in \mathbb{Z}$

$\Theta = 2z = \frac{\pi}{2}(2n+1) \Rightarrow z_0 = z_n = \frac{\pi}{4}(2n+1)$, $n \in \mathbb{Z}$

$$\boxed{\text{Res}\left(\tan(2z)\,;\,z_n\right) = \frac{\sin(2z_n)}{-2\sin(2z_n)} = \frac{-1}{2} \quad \forall n \in \mathbb{Z}.}$$

Notice, $\sin(2z_n) = \sin\left(\frac{\pi}{2}(2n+1)\right) = \pm 1 \neq 0$ hence the PQ-f'la applied.

$$\boxed{\text{Th}^{\underline{m}}/ \text{ If } f \text{ has pole of order } m \text{ at } z_0 \text{ then } \quad \text{Res}(f\,;\,z_0) = \lim_{z \to z_0}\left[\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\right]}$$

<u>Proof</u>: left to reader for now. It's just an explicit Laurent series computation.

$\boxed{E107}$ Let $f(z) = \dfrac{\sin(z)}{(z-1)^3}$ calculate $\mathrm{Res}(1)$.

Clearly $f(z)$ has pole of order 3 at $z_0 = 1$.
Apply the Th$^{\underline{m}}$ with $m=3$, $z_0 = 1$,

$$\frac{d^2}{dz^2}\left[(z-1)^3 f(z)\right] = \frac{d^2}{dz^2}\left[\sin(z)\right] = -\sin(z).$$

$$\therefore \mathrm{Res}(f;1) = \lim_{z\to 1}\left[\frac{1}{2!}(-\sin(z))\right] = \boxed{\frac{-\sin(1)}{2}}$$

$\boxed{E108}$ Alternatively, w/o Th$^{\underline{m}}$, not as easy,

$$\sin(z) = \sin(z-1+1)$$
$$\sin(z) = \sin(z-1)\cos(1) + \cos(z-1)\sin(1)$$

when divide by $(z-1)^3$ the $\cos(z-1) = 1 - \frac{1}{2}(z-1)^2 + \cdots$ gives the $j=-1$ coeff and so we pick-up the $-\frac{\sin(1)}{2}$.

In §56 of Churchill, he presents this Th$^{\underline{m}}$ as follows

Th$^{\underline{m}}$/ If $f(z)$ has isolated singularity which is a pole of order m then $f(z)$ can be written $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic and non zero at z_0. Moreover,

① $\displaystyle \mathrm{Res}_{z=z_0}(f(z)) = \mathrm{Res}(f(z), z_0) = \phi(z_0)$ $(m=1)$

② $\displaystyle \mathrm{Res}_{z=z_0}(f(z)) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if $m \geq 2$

There are 4 additional, <u>nice</u> examples on how to use this Th$^{\underline{m}}$ on pg. 192–193 of Churchill.

- further calculation of Laurent series is accomplished by applying the PQ – f'la and its m^{th} order generalization along side the direct calculational techniques we began studying a week or so ago. I turn to integration techniques.

$$\boxed{\text{E109}} \quad I = \int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\, d\theta$$

We looked at another of these previously. I include this for a 2^{nd} look at the idea

$$z = e^{i\theta}$$
$$dz = ie^{i\theta}\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$
$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

Let's identify I as $\oint_C f(z)dz$ for $C: z = e^{i\theta},\ 0 \le \theta \le 2\pi$,

$$\frac{\sin^2\theta}{5+4\cos\theta}\,d\theta = \frac{\frac{-1}{4}\left(z-\frac{1}{z}\right)^2}{5+2\left(z+\frac{1}{z}\right)}\left(\frac{dz}{iz}\right)$$
$$= \frac{-1}{4i}\left(\frac{z^2 - 2 + 1/z^2}{5z + 2z^2 + 2}\right)dz$$
$$= \frac{-1}{4i}\left[\frac{z^4 - 2z^2 + 1}{z^2(2z^2+5z+2)}\right]dz$$
$$= \frac{-1}{4i}\left[\frac{(z^2-1)^2}{z^2(2z+1)(z+2)}\right]dz$$

$$I = \oint_C \frac{-1}{4i}\underbrace{\left[\frac{(z^2-1)^2\,dz}{2z^2(z+1/2)(z+2)}\right]}_{f(z)} = \frac{-1}{4i}(2\pi i)\left[\operatorname{Res}(f; \tfrac{-1}{2}) + \operatorname{Res}(f;0)\right]$$
$$= \frac{-\pi}{2}\left[\frac{3}{4} - \frac{5}{4}\right] \quad \text{details}$$
$$= \boxed{\frac{\pi}{4}}$$

E109 continued

$$\operatorname*{Res}_{z=\frac{-1}{2}}\left(\frac{(z^2-1)^2}{2z^2(z+1/2)(z+2)}\right) = \underbrace{\frac{z^2-1}{2z^2(z+2)}}_{\phi(z)}\Bigg|_{z=-1/2} = \frac{\left(\frac{1}{4}-1\right)^2}{2\left(\frac{1}{4}\right)\left(-\frac{1}{2}+2\right)}$$

$\phi(z)$ described by Churchill.

$$= \frac{9/16}{\frac{1}{2}\left(\frac{3}{2}\right)} = \frac{3}{16/4} = \frac{3}{4}.$$

$$\phi(z) = \frac{(z^2-1)^2}{(2z+1)(z+2)}$$

$$\operatorname*{Res}_{z=0}\left(\frac{\phi(z)}{z^2}\right) = \lim_{z\to 0}\left[\frac{1}{1!}\frac{d}{dz}\left[\frac{(z^2-1)^2}{2z^2+5z+2}\right]\right]$$

$$= \lim_{z\to 0}\left(\frac{2(z^2-1)(2z)[2z^2+5z+2]-(z^2-1)^2[4z+5]}{(2z^2+5z+2)^2}\right)$$

$$= \frac{-5}{4}.$$

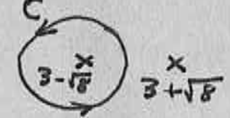
$\boxed{\text{E110}}$ Show $\displaystyle\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta} = \pi\sqrt{2}$

Remark: the w-substitution below is due to Gmny. Direct computation in z is uglier.

$$I = \int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta} = \int_C\left[\frac{dz/iz}{1-\frac{1}{4}(z-z^{-1})^2}\right]\frac{4z}{4z} = \int_C\frac{-4iz\,dz}{4z^2\left[1-\frac{1}{4}(z^2-2+1/z^2)\right]} = \int_C\frac{-4iz\,dz}{4z^2-z^4+2z^2-1}$$

Thus, $I = \int_C \frac{4iz\,dz}{z^4-6z^2+1}$. Make a $w = z^2$ substitution. Observe that $w = z^2 \Rightarrow dw = 2z\,dz$ and $z = e^{i\theta} \Rightarrow z^2 = e^{2i\theta} \hookrightarrow C$ gets covered twice in the w-variable, we'll denote twice wound by $2C$,

$$I = \int_{2C}\frac{2i\,dw}{w^2-6w+1} = \int_{2C}\frac{2i\,dw}{(w-3+\sqrt{8})(w-3-\sqrt{8})}$$



$$= 2(2\pi i)\operatorname{Res}\left(\frac{2i}{w^2-6w+1};\ 3-\sqrt{8}\right)$$

$$= \frac{8\pi i^2}{w-3-\sqrt{8}}\Bigg|_{w=3-\sqrt{8}}$$

$$= \frac{-8\pi}{3-\sqrt{8}-3-\sqrt{8}} = \frac{-8\pi}{-2\sqrt{8}} = \frac{4\pi}{2\sqrt{2}} = \boxed{\pi\sqrt{2}}$$

$\boxed{E111}$ $\displaystyle\int_0^{2\pi} \frac{d\theta}{2-\cos\theta}$ calculate.

Let $z = e^{i\theta}$ then $dz = ie^{i\theta}d\theta \hookrightarrow d\theta = dz/iz$
and $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + z^{-1})$ hence,

$$\int_0^{2\pi} \frac{d\theta}{2-\cos\theta} = \int_C \frac{dz/iz}{2 - \frac{1}{2}(z+z^{-1})}$$

$$= \frac{2}{i}\int_C \frac{dz}{z(4-(z+z^{-1}))}$$

$$= \frac{2}{i}\int_C \frac{dz}{4z - z^2 - 1}$$

$$= 2i\int_C \frac{dz}{z^2-4z+1}$$

$$= 2i\int_C \frac{dz}{(z-2-\sqrt{3})\underbrace{(z-2+\sqrt{3})}}$$

$z = 2-\sqrt{3} \in$ interior (C).

$$= 2i\left[2\pi i \, \mathrm{Res}\left(\frac{1}{z^2-4z+1};\ 2-\sqrt{3}\right)\right]$$

$$= -4\pi\left(\frac{1}{z-2-\sqrt{3}}\bigg|_{2-\sqrt{3}}\right)$$

$$= -4\pi\left(\frac{1}{\cancel{2}-\sqrt{3}-\cancel{2}-\sqrt{3}}\right)$$

$$= \boxed{\frac{2\pi}{\sqrt{3}}}$$

Remark: $\displaystyle\int_0^{\pi} \frac{d\theta}{2-\cos\theta}$ can also be computed as above if we note $\cos(\theta) = \cos(2\pi - \theta)$ hence $\displaystyle\int_0^{\pi}\frac{d\theta}{2-\cos\theta} = \int_{\pi}^{2\pi}\frac{d\theta}{2-\cos\theta}$ (by $u = 2\pi - \theta$, $du = -d\theta$, $u(0) = 2\pi$, $u(\pi) = \pi$; then flip bounds and $u \to \theta$ at end.)

Therefore

$$\int_0^{\pi}\frac{d\theta}{2-\cos\theta} = \frac{1}{2}\int_0^{2\pi}\frac{d\theta}{2-\cos\theta} = \frac{\pi}{\sqrt{3}}.$$

not generally true!

Principal Values and Improper Integrals

Recall from calculus II,

$$\int_0^\infty f(x)\,dx = \lim_{t\to\infty}\int_0^t f(x)\,dx$$

$$\int_{-\infty}^0 f(x)\,dx = \lim_{s\to-\infty}\int_s^0 f(x)\,dx$$

$$\underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{\text{exists provided these}} = \underbrace{\int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx}_{\text{exist.}}$$

When the $\int_0^\infty$, $\int_{-\infty}^0$ both exist then $\int_{-\infty}^{\infty} f(x)dx$ can also be calculated as

$$\int_{-\infty}^{\infty} f(x)\,dx = \boxed{\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx \overset{\text{def}}{\equiv} P.V.\int_{-\infty}^{\infty} f(x)\,dx}$$

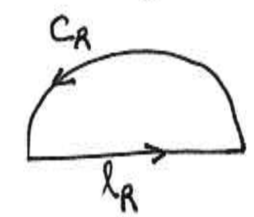
However be careful, it may happen that the principal value exists whereas $\int_{-\infty}^{\infty}$ itself diverges...

$\boxed{E112}$ $P.V.\int_{-\infty}^{\infty} f(x)\,dx = 0$ if we're given $f(-x) = -f(x)$ since $\int_{-R}^{R} f(x)dx = 0$ for any odd fnct. Of course, $\int_0^\infty f(x)dx$ need not exist for an arbitrary odd fnct.!

Typical Strategy: to calculate $P.V.\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}dx$

1.) consider extending $f(z) = \frac{p(z)}{q(z)}$ where $q(z)$ has no real zeros (unless $p(z)$ cancels them)

2.) If $\deg(p) \leq \deg(q) - 2$ then



$$\underbrace{\int_{H_R} f(z)\,dz}_{\substack{\text{can calculate}\\ \text{by residue theorem}}} = \underbrace{\int_{l_R} f(z)\,dz}_{\substack{\text{gives P.V.}\\ \text{as } R\to\infty}} + \underbrace{\int_{C_R} f(z)\,dz}_{\substack{\text{can be shown}\\ \text{to} \to 0\\ \text{by bounding}\\ \text{theorem.}}}$$

$\boxed{E113}$ Find P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$ (#1 of Saff & Snider's §6.3)

Let $f(z) = \dfrac{1}{z^2+2z+2} = \dfrac{1}{(z+1-i)(z+1+i)}$

the only singularity of $f(z)$ in the upper-half-plane is at $z_0 = -1+i$ and we can calculate

$$\operatorname{Res}(f(z); z_0) = \left.\frac{1}{z+1+i}\right|_{z=z_0=-1+i} = \frac{1}{-1+i+1+i} = \frac{1}{2i}$$

Let H_R be the closed half-circle, $H_R = l_R \cup C_R$

By Cauchy's Res. theorem,

$$\int_{H_R} f(z)\,dz = \frac{2\pi i}{2i} = \pi.$$

Furthermore, note for $z = Re^{i\theta}$ and $R > 2$,

$$\left| \frac{1}{z^2+2z+2} \right| = \frac{1}{|(z+1)^2+1|} \leq \frac{1}{\big||z+1|^2-1\big|} \leq \frac{1}{|(|z|-1)^2-1|}$$

Hence,

$$|f(z)| \leq \frac{1}{(R-1)^2-1} = \frac{1}{R^2-2R+1-1} = \frac{1}{R^2-2R}$$

By bounding theorem,

$$\left| \int_{C_R} f(z)\,dz \right| \leq \frac{\pi R}{R^2-2R} \longrightarrow 0 \text{ as } R \to \infty.$$

Thus,

$$\lim_{R\to\infty} \left(\int_{H_R} f(z)\,dz \right) = \lim_{R\to\infty} \left(\int_{l_R} f(z)\,dz + \underbrace{\cancel{\int_{C_R} f(z)\,dz}}_{\to 0} \right)$$

$$\Rightarrow \quad \pi = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2+2x+2}$$

$$\therefore \quad \boxed{\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \pi}$$

the bounding argument of $\boxed{E113}$ we've seen before and it's useful to argue it in general to save our effort in future.

Th^{m}/ If $C_R : z = Re^{i\theta}$ for $[\theta_1, \theta_2]$ and $f(z) = P(z)/Q(z)$ with $\deg(Q) \geq 2 + \deg(P)$ for rational fncts. $P(z)$ and $Q(z)$ then $\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$

<u>Proof</u>: essentially college algebra.

$$f(z) = \frac{a_0 + a_1 z + \cdots + a_m z^m}{b_0 + b_1 z + \cdots + b_n z^n} = \frac{(a_0/z^m + a_1/z^{m-1} + \cdots + a_m)z^m}{(b_0/z^n + b_1/z^{n-1} + \cdots + b_n)z^n}$$

thus, as $z = Re^{i\theta}$,

$$|f(z)| = R^{m-n} \frac{|a_0/z^m + a_1/z^{m-1} + \cdots + a_m|}{|b_0/z^n + b_1/z^{n-1} + \cdots + b_n|}$$

$$\leq R^{m-n}\left(\frac{|a_0|/R^m + |a_1|/R^{m-1} + \cdots + |a_m|}{\big|\,|b_0/z^n + \cdots + b_{n-1}/z| - |b_n|\,\big|}\right) \quad R > 1.$$

$$\leq R^{m-n}\left(\frac{|a_0| + |a_1| + \cdots + |a_m|}{\big|\,\big||b_0/z^n + \cdots + b_{n-2}/z^2| - b_{n-1}/R\big| - |b_n|\,\big|}\right)$$

$$\vdots$$

$$\leq R^{m-n}\left(\frac{|a_0| + |a_1| + \cdots + |a_m|}{\underbrace{\big|\big|\big|\cdots\big||b_0/R^n| - |b_1|/R^{n-1}\big| - \cdots - |b_{n-1}/R|\big| - |b_n|\big|}_{\text{little sketchy here, but you can argue this is bounded by some } M > 0}}\right)$$

thus $|f(z)| \leq R^{m-n} M$

Hence, by bounding theorem,

$$\left|\int_{C_R} f(z)dz\right| \leq R(\theta_2 - \theta_1) R^{m-n} M = M(\theta_2 - \theta_1) R^{m-n+1}$$

but we assumed $\underbrace{n \geq 2 + m}_{-2 \geq m-n}$ $\Rightarrow m - n + 1 \leq -1$ hence the expression above $\to 0$ as $R \to \infty$.

Comment: to avoid the sketchy step, Saff and Snider noted that as $R \to \infty$

$$|f(z)| = R^{m-n} \frac{|a_0/z^m + \cdots + a_m|}{|b_0/z^n + \cdots + b_n|} \longrightarrow 0$$

thus for $|z|$ sufficiently large it must be the case that: (let's say for $|z| > N$)

$$\frac{|a_0/z^m + \cdots + a_m|}{|b_0/z^n + \cdots + b_n|} < \frac{|a_m|}{|b_n|} + 1$$

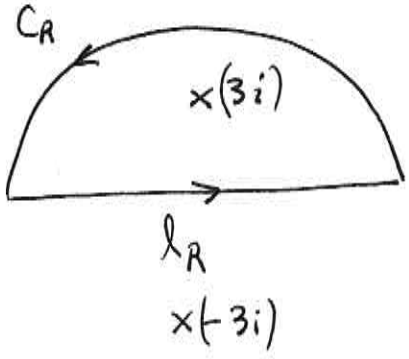
Hence as $R \to \infty$, certainly $|z| = R > N$ so the bounding th^{m} applies,

$$\left|\int_{C_R} f(z)\,dz\right| \leq (\theta_2 - \theta_1) R \left[\frac{|a_m|}{|b_n|} + 1\right] R^{m-n}.$$

Remark: it's nice that this Th^{m} applies to any arc which tends to ∞.

Sometimes we need to consider the lower half-plane or smaller arcs. We'll see ...

E114 Calculate P.V. $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+9)^2}$



$$f(z) = \frac{z^2}{(z^2+9)^2}$$

singularity at $z = \pm 3i$

H_R encloses for $R > 3$.

$$\operatorname*{Res}_{z=3i} \left(\frac{z^2}{(z+3i)^2(z-3i)^2} \right) = \frac{d}{dz}\left[\frac{z^2}{(z+3i)^2} \right] \Bigg|_{z=3i}$$

$$= \frac{2z(z+3i)^2 - z^2(2(z+3i))}{(z+3i)^4} \Bigg|_{z=3i}$$

$$= \frac{2z(z+3i) - 2z^2}{(z+3i)^3}$$

$$= \frac{6i(6i) - 2(-9)}{(6i)^3}$$

$$= \frac{-18}{6(36)(-i)} = \frac{3}{36i}$$

Thus, $\int_{H_R} f(z)\,dz = 2\pi i \left(\frac{-i}{12}\right) = \pi/6$ for $R > 3$.

Note P.V. $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+9)^2} = \lim_{R\to\infty} \int_{l_R} f(z)\,dz$. And,

$$\lim_{R\to\infty} \int_{H_R} f(z)\,dz = \lim_{R\to\infty} \int_{l_R} f(z)\,dz + \cancelto{0}{\lim_{R\to\infty} \int_{C_R} f(z)\,dz}$$ by the just proved.

$$\therefore \boxed{\text{P.V.} \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+9)^2} = \pi/6}$$

$\boxed{E115}$ Calculate P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}\,dx}{1+e^x}$ for $0<a<1$

$e^{az} \neq 0$ for all $z \in \mathbb{C}$. In fact e^{az}, $1+e^z$ are entire. It follows that the zeros of $1+e^x$ give simple poles. There are ∞ly many of them. (Example 3 of Saff & Snider pg. 324)

Recall $e^z = e^{z+2\pi i k}$ for $k \in \mathbb{Z}$. Note

$$e^{\pi i} = \cos\pi + i\sin\pi = -1 \quad \therefore \quad e^{\pi i}+1 = 0.$$

We face simple poles for $f(z) = \dfrac{e^{az}}{1+e^z}$ at

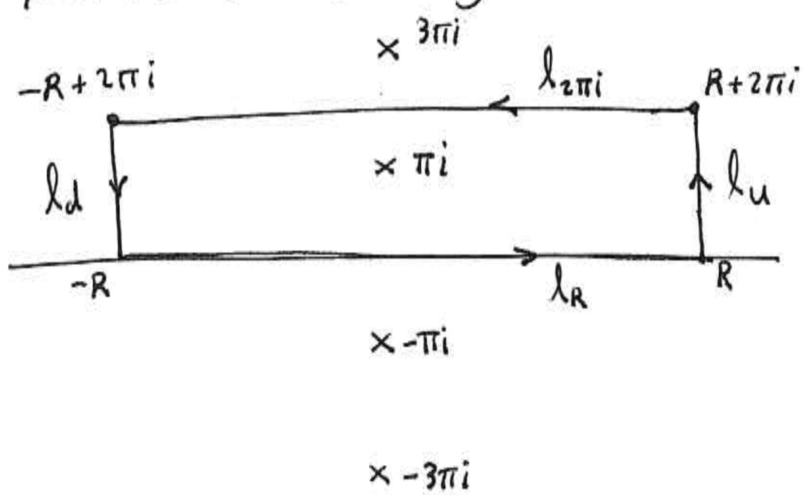
$$z_k = \pi i(1+2k), \quad k \in \mathbb{Z}.$$

$$\text{Res}(f(z); z_k) = \frac{e^{az_k}}{e^{z_k}}$$

$$\Rightarrow \underline{\text{Res}(f(z); z_k) = e^{(a-1)z_k}}$$

[by the PQ-f'la $P(z) = e^{az}$, $Q(z) = 1+e^z$

Let's picture these singularities.



$$\Gamma_R = \underbrace{l_R}_{\text{gives P.V.}} \cup l_u \cup \underbrace{l_{2\pi i}}_{\text{intuitively related to } l_R \text{ somehow}} \cup l_d$$

expect to vanish (l_u, l_d)

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i \,\text{Res}(f(z); \pi i) = 2\pi i\, e^{(a-1)\pi i}$$

$$\int_{\Gamma_R} f(z)\,dz = \int_{l_R} f + \int_{l_{2\pi i}} f + \overbrace{\int_{l_u} f + \int_{l_d} f}^{\text{show these vanish as } R\to\infty \text{ next}}$$

E115 continued:

$$\left|\int_{l_u} \frac{e^{az}\,dz}{1+e^z}\right| = \left|\int_0^1 \left(\frac{e^{a(R+2\pi it)}}{1+e^{R+2\pi it}}\right)(2\pi i\,dt)\right| \qquad z = R + it(2\pi) \text{ for } 0 \le t \le 1$$

$$\le \int_0^1 \frac{|e^{aR}||e^{a(2\pi it)}|}{||1| - |e^R e^{2\pi it}||} 2\pi\,dt$$

$$= \int_0^1 \left(\frac{e^{aR}}{1-e^R}\right) 2\pi dt$$

$$= \frac{2\pi e^{aR}}{1-e^R} \xrightarrow[R\to\infty]{\mathcal{L}\ (\frac{\infty}{\infty})} \frac{2\pi a e^{aR}}{-e^R} = 2\pi a e^{R(a-1)} \longrightarrow 0 \quad (\text{given } 0<a<1)$$

Thus, as $R \to \infty$ we find $\lim_{R\to\infty}\left(\int_{l_{R}} f(z)dz\right) = 0$.

Likewise, you can show $\lim_{R\to\infty} \int_{l_d} f(z)dz = 0$. Consider,

$$\int_{l_{2\pi i}} \frac{e^{az}\,dz}{1+e^z} = \int_{-R}^{R} \frac{e^{a(2\pi i - t)}(-dt)}{1+e^{2\pi i - t}} \qquad \left[\begin{array}{l} z(t) = 2\pi i - t \\ z(-R) = 2\pi i + R \\ z(R) = 2\pi i - R \\ dz = -dt \end{array}\right.$$

$$= \int_{-R}^{R} \frac{e^{2\pi ia}\, e^{-at}(-dt)}{1+e^{-t}}$$

$$= e^{2\pi ia} \int_R^{-R} \frac{e^{au}\,du}{1+e^u} \qquad \left[\begin{array}{l} u = -t \\ -at = au \\ du = -dt \\ u(R) = -R,\ u(-R) = R \end{array}\right.$$

$$= -e^{2\pi ia} \int_{-R}^{R} \frac{e^{au}\,du}{1+e^u}$$

$$= -e^{2\pi ia} \int_{l_R} f(z)dz$$

Thus,

$$\int_{\Gamma_R} f(z)dz = \int_{l_R} f(z)dz - e^{2\pi ia}\int_{l_R} f(z)dz + \int_{l_u \cup l_d} f(z)dz$$

As $R \to \infty$ we obtain

$$2\pi i e^{(a-1)\pi i} = (1 - e^{2\pi ia}) \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ax}\,dx}{1+e^x}$$

$$P.V. \int_{-\infty}^{\infty} \frac{e^{ax}\,dx}{1+e^x} = \frac{2\pi i e^{a\pi i}}{e^{2\pi ia} - 1} = \frac{2\pi i}{e^{\pi ia} - e^{-\pi ia}} = \boxed{\frac{\pi}{\sin(a\pi)}}.$$

Integrals Involving sine, cosine from $-\infty \to \infty$
principle value calculations

$\boxed{\text{E116}}$ Calculate $\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos 3x}{x^4+4}\, dx$ following Saff & Snider pg. 329

Let $f(z) = \frac{\cos 3z}{z^4+4}$. Recall

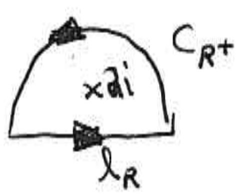
$$\cos(3z) = \frac{1}{2}\left(e^{3iz} + e^{-3iz}\right)$$

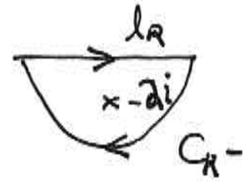
$$\Rightarrow \cos(3(x+iy)) = \underbrace{\frac{1}{2} e^{3ix} e^{-3y}}_{\text{blows-up in } \mathbb{H}_+} + \underbrace{\frac{1}{2} e^{-3ix} e^{3y}}_{\text{blows-up in } \mathbb{H}_-}$$

$\text{Im}(z) \geq 0$. $\mathbb{H}_+$

$\mathbb{H}_-$ $\text{Im}(z) \leq 0$

Take apart $I = \text{P.V.} \int_{-\infty}^{\infty} \frac{\cos 3x}{x^2+4}\, dx$ into two parts.





$$I_1 = \int_{-R}^{R} \frac{e^{3ix}}{2(x^4+4)}\, dx + \int_{C_{R^+}} \frac{e^{3iz}\, dz}{2(z^4+4)} = 2\pi i \operatorname*{Res}_{z=2i}\left(\frac{e^{3iz}}{2(z^2+4)}\right) = 2\pi i \left(\frac{e^{-6}}{8i}\right) = \frac{\pi e^{-6}}{4}$$

$$I_2 = \int_{-R}^{R} \frac{e^{-3ix}}{2(x^4+4)}\, dx + \int_{C_{R^-}} \frac{e^{-3iz}\, dz}{2(z^4+4)} = -2\pi i \operatorname*{Res}_{z=-2i}\left(\frac{e^{-3iz}}{2(z^2+4)}\right) = -2\pi i \left(\frac{-e^{-6}}{8i}\right) = \frac{\pi e^{-6}}{4}$$

$$I = \lim_{R\to\infty}(I_1 + I_2) = \boxed{\frac{\pi e^{-6}}{2}}$$

Since C_{R^+}, C_{R^-} are trivial as $R \to \infty$ by an argument as follows:

$$\left|\frac{e^{3iz}}{2(z^4+4)}\right| \leq \frac{e^{-3y}}{2|R^4-4|} \longrightarrow 0 \quad \text{as } R \to \infty \text{ and } x+iy \in C_{R^+}$$

Likewise $\left|\frac{e^{-3iz}}{2(z^4+4)}\right| \leq \frac{e^{3y}}{2|R^4-4|} \longrightarrow 0$ as $R \to \infty$, $x+iy \in C_{R^-}$

$\boxed{E116}$ Continued, clearly for $R \gg 0$ we find $M_\pm$ such that $e^{\pm y} < M_\pm$ and

$$\int_{C_R^\pm} \frac{e^{3iz}\,dz}{2(z^4+4)} \leq \frac{M_\pm \pi R}{2(R^4-4)} \longrightarrow 0$$

as $R \longrightarrow \infty$.

Jordan's Lemma:

If $\deg(Q) \geq 1 + \deg(P)$ for polynomial P, Q then
$$\lim_{R\to\infty} \int_{C_R^+} e^{imz} \frac{P(z)}{Q(z)}\,dz = 0$$
where $C_R^+ = \{ z \in \mathbb{C} \mid |z| = R \}$

Proof: follows from explicit calculation with parameterization of C_R^+

$\boxed{E117}$ $P.V.\int_{-\infty}^{\infty} \frac{2\cos x}{x+i}\,dx$

$$f(z) = \frac{2\cos z}{z+i} = \frac{e^{iz}+e^{-iz}}{z+i} = f_1(z) + f_2(z)$$

$$P.V.\int_{-\infty}^{\infty} \frac{2\cos x\,dx}{x+i} = P.V.\int_{-\infty}^{\infty} \frac{e^{ix}}{x+i} + P.V.\int_{-\infty}^{\infty} \frac{e^{-ix}}{x+i}$$
$$= \cancel{2\pi i \operatorname{Res}_{z=i}\left(\frac{e^{iz}}{z+i}\right)} - 2\pi i \operatorname{Res}_{z=-i}\left(\frac{e^{-iz}}{z+i}\right)$$
$$= \cancel{2\pi i\, e^{z^2}} - 2\pi i\, e^{+z^2}$$

think.

$$= \boxed{-2\pi i/e}$$

how?

Remark: $\boxed{E117}$ is how not to write your work we'll repair this in lecture.