PRIMES & HIS FAVORITE THINGS (ANALYSIS & # theory)

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$\log(x)$ = natural log

$$f(x) \sim g(x) \iff \left|\frac{f(x)-g(x)}{g(x)}\right| \longrightarrow 0 \quad \text{as } x \longrightarrow \infty$$

$g(x) = O(f(x)) \iff \exists\, c>0$ s.t. $|g(x)| < c f(x)$.

for (5)

▶ Euclid (c.a. 300BC)

- Fundamental Th^m of Arithmetic
  - ▶ every pos. integer can be written as a product of primes in a unique way. (well, upto reordering so if we impose an order then unique.)

$$n = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n} \quad \leftarrow \text{prime-power factorization}$$

CONCEPT: primes are atomic.

- Th^m/ There are infinitely many primes because any finite list of primes is incomplete. (follows from FTA by slick argument ~~for~~)

▶ Euler (AD 1737) : maybe most creative mathematician that ever lived.

Th^m/ There are ∞ly many primes because

$$\sum_{p \text{ prime}} \frac{1}{p} = \infty$$

Euler Writes: "$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = \log(\log(\infty))$"

Proof: $\sum_{n=1}^{\infty} \frac{1}{n^s}$ define $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges for $s>1$ and diverges at $s=1$ by $\int$-test.

an s-series since p is reserved to mean prime in sequel

By FTA,
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$
$$= \prod_p \left(\sum_{k=0}^{\infty} \frac{1}{p^{ks}} \right) =$$
$$= \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \cdots\right)\left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \cdots\right)$$
$$= \prod_p \frac{1}{1 - 1/p^s}$$
$$= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \quad \text{for } s>0$$

formally take log, following Euler, for $s > 1$

$$\log(\zeta(s)) = -\sum_p \log\left(1 - \frac{1}{p^s}\right)$$

$$= -\left(\sum_p \sum_{n=1}^{\infty} \frac{-1}{n\, p^{ns}}\right)$$

$$= \sum_p \frac{1}{p^s} + \sum_p \sum_{n=2}^{\infty} \frac{1}{p^{ns}} \qquad (*)$$

For $s > 1$,

$$\sum_p \sum_{n=2}^{\infty} \frac{1}{n\, p^{ns}} \leq \sum_p \sum_{n=2}^{\infty} \frac{1}{p^n} = \sum_p \frac{1}{p^2} \sum_{m \geq 0} \frac{1}{p^m}$$

$$= \sum_p \frac{1}{p^2}\left(\frac{1}{1-\frac{1}{p}}\right)$$

$$= \sum \frac{1}{p(p-1)}$$

$$\leq \sum \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$$

Take ~~to~~ the limit as $s \to 1^+$ both sides of $*$

$$\lim_{s \to 1^+} \log(\zeta(s)) = \infty \quad \text{and} \quad \sum_p \sum_{n=2}^{\infty} \frac{1}{n\, p^{ns}} < \infty$$

$$\Rightarrow \lim_{s \to 1^+} \left(\sum_p \frac{1}{p^s}\right) = +\infty \quad //$$

Remark: this proof opens the door to using calculus/analysis to study # theory.

Gauss (AD 1792 or 1793) (he was 15 or 16)

CONJECTURE: Given an integer t the "probability" that t is prime is $\frac{1}{\log t}$

Prime # Th^m In Rudimentary Form.

We interpret this to mean

$$\pi(x) = \#\{p \le x : p \text{ prime}\} \sim \int_2^x \frac{dt}{\log t} = Li(x)$$

Gauss explitly calculates the error in $\pi(x)$ by $Li(x)$ was small.

- no progress for about 70 years.

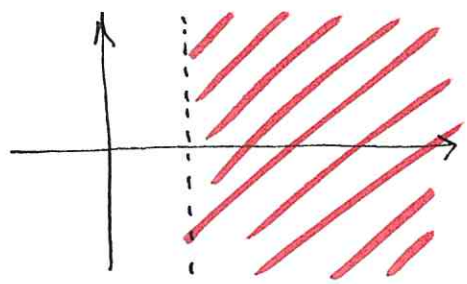
Riemann (AD 1859/1860): Gives an 8 page address on prime # theory: lays out plan for proving the P#T.

summary: "the zeros of $\zeta(s)$ know everything about the primes"

- Where have all the zeros gone?

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } s > 1$$

... HOW CAN THIS BE ZERO!? $\sum$ pos. #'s.



▸ formula above, good for Real (s) > 1.

- The Euler Product $\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \Rightarrow$ still no zeros.

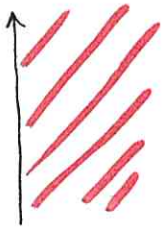
- we need to go on a quest to find zeros.

▸ How do we analytically continue a function?
↳ By all means necessary!

Euler studied the funct.

$$\Gamma(s) = \int_0^\infty t^s e^{-t} \frac{dt}{t}$$ (Riemann would have noted works for $Re(s) > 0$)

in a quest to find a smooth interpolation of $n!$ Lets play with Γ a bit.



$$\Gamma(s) = \int_0^\infty t^s e^{-t} \frac{dt}{t}$$

$= \text{well} \ldots$

I.B.P. on $\Gamma(s+1) = \int_0^\infty t^s e^{-t} dt \leftarrow \begin{bmatrix} u = t^s & dv = e^{-t} dt \\ du = st^{s-1} & v = -e^{-t} \end{bmatrix}$

$$= -e^{-t} t^s \Big|_0^\infty + s \int_0^\infty t^{s-1} e^{-t} dt$$

$$= s \int_0^\infty t^s e^{-t} \frac{dt}{t}$$

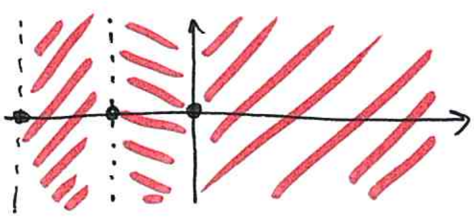
$$= s\Gamma(s).$$

(Euler probably thought of s as real)

But, $\Gamma(1) = \int_0^\infty e^{-t} dt = 1 \Rightarrow \Gamma(n+1) = n!$

Notice,

$$\Gamma(s) = \frac{\Gamma(s+1)}{s} = \underbrace{\frac{1}{s} \int_0^\infty t^{s+1} e^{-t} \frac{dt}{t}}_{\text{makes sense for } Re(s) > -1,\ s \neq 0}$$



extend via the integral f-la each strip.

$s = 0, s = -1, s = -2, \ldots$

each a simple pole.

$\Rightarrow$ meromorphic extension.

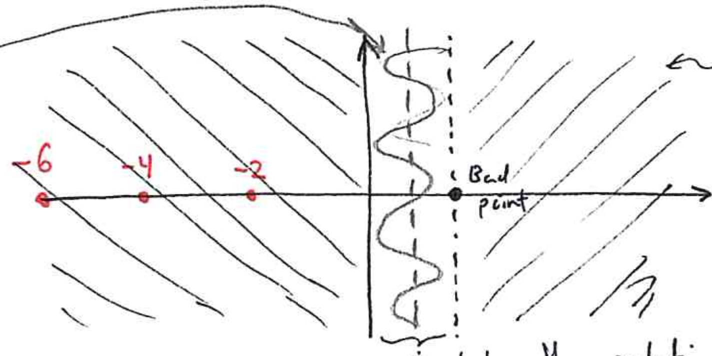
Riemann proved in 1859 ~ 1860 or so...

The Zeta fnct. can be analytically continued to the whole complex plane $\mathbb{C}$ except for a simple pole of residue 1. Furthermore,

$$(*) \quad \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) \quad \text{for all } s \in \mathbb{C} - \{1\}.$$

critical strip. $0 < Re(s) < 1$

critical line: $Re(s) = \frac{1}{2}$



Here $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)$

$\Gamma(\frac{s}{2})$'s poles $\Rightarrow \zeta(s)$ has zeros at $s = -2n$ for $n = 1, 2, 3, \ldots$

missed by the relation * above.

In addition Riemann conjectures

1. $\zeta(s)$ has ∞ly many zeros in the critical strip
2. even gives an approximate formula for the # of nontrivial zeros.
3. gives an explicit formula relating prime #'s to zeros of $\zeta(s)$

Notation: $$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k \\ 0 & \text{otherwise} \end{cases}$$

Von Mangoldt (AP 1895)

Thm: $$\sum_{p^k \le x}' \log(p) = \sum_{n \le x}' \Lambda(n) = x - \frac{\zeta'(0)}{\zeta(0)} - \sum_{\substack{\rho \\ \xi(\rho)=0}}' \frac{x^\rho}{\rho}$$

Btw: $\underbrace{\zeta(0) = \frac{-1}{2}}_{\text{easy}}$ and $\underbrace{\frac{\zeta'(0)}{\zeta(0)} = \log(2\pi)}_{\text{not so easy -- Hadamard Products.}}$

4. "It seems very likely that all of the nontrivial zeros of $\zeta(s)$ lie on the line $Re(s) = 1/2$" (Riemann Hypothesis)

$$\#\{p \le x\} = \int_2^x \frac{dt}{\log(t)} + O\left(\sqrt{x}\log(x)\right)$$

Sketch Proof: Start with
$$\frac{1}{2\pi i} \int_{Re(s)=2} \frac{y^s}{s}\, ds = \begin{cases} 0 & \text{if } 0 < y < 1 \\ 1/2 & \text{if } y = 1/2 \\ 1 & \text{if } y > 1 \end{cases}$$

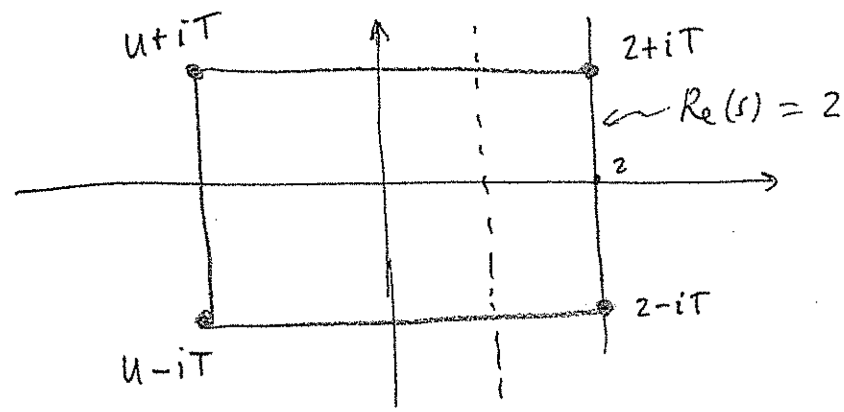
And note that
$$\frac{\zeta'(s)}{\zeta(s)} = \frac{d}{ds}\left[\log(\zeta(s))\right) = \frac{-d}{ds} \sum_p' \log\left(1 - \frac{1}{p^s}\right)$$
$$= -\sum_p' \frac{1}{1 - \frac{1}{p^s}} \left(-\log\left(\frac{1}{p}\right) \frac{1}{p^s}\right)$$
$$= -\sum_p' \frac{\log(p)}{p^s} \sum_{k=0}^{\infty}' \frac{1}{p^{ks}}$$
$$= \sum_p' \sum_{n=1}^{\infty}' \frac{\log(p)}{p^{ks}} = -\sum_{n=1}^{\infty}' \frac{\Lambda(n)}{n^s}$$

If $x \notin \mathbb{Z}$

$$\sum_{n\le x}{}' \Lambda(n) = \frac{-1}{2\pi i}\int_{Re(s)=2} \sum_{n\le x}{}' \frac{\Lambda(n)}{n^s}\,\frac{x^s}{s}\,ds$$

$$= \frac{-1}{2\pi i}\int_{Re(s)=2}\sum_{n=1}^{\infty}{}' \frac{\Lambda(n)}{n^s}\,\frac{x^s}{s}\,ds = \frac{-1}{2\pi i}\int_{Re(s)=2}\frac{\zeta'(s)}{\zeta(s)}\,\frac{x^s}{s}\,ds$$

$$= \frac{-1}{2\pi i}\int_{2-iT}^{2+iT}\frac{\zeta'(s)}{\zeta(s)}\,\frac{x^s}{s}\,ds + E_1(T,x)$$ where $E_1(T,x)\to 0$ as $T\to\infty$



$$= \frac{-1}{2\pi i}\int_{R(T,u)}\frac{\zeta'(s)}{\zeta(s)}\,\frac{x^s}{s}\,ds + \underbrace{E_1(T,x)}_{\substack{\to 0\\ T\to\infty}} + \underbrace{E_2(T,u,x)}_{\to 0 \text{ as } u\to -\infty}$$

$$= x - \frac{\zeta'(0)}{\zeta(0)} - \sum_{\rho}{}' \frac{x^\rho}{\rho} + \underset{\to 0}{E} + \underset{\to 0}{E}$$ need to calculate Residues inside the rectangle.

(Von Mangoldt's Result! (50 yrs after Riemann to get it))

<u>Returning to early eqn</u>

$$\sum_{p^k\le x}\log(p) = \sum_{n\le x}{}'\Lambda(n) = x - \frac{\zeta'(0)}{\zeta(0)} - \sum_{\rho:\,\zeta(\rho)=0}{}' \frac{x^\rho}{\rho}$$

$$= x - \log(2\pi) - \sum_{n=1}^{\infty}\frac{x^{-2n}}{2n} - \sum_{\rho \text{ nontriv}}\frac{x^\rho}{\rho}$$

$$= x - \log(2\pi) - \frac{1}{2}\log(1-x^{-2}) - \sum_{\text{nontriv}}{}'\frac{x^\rho}{\rho}$$