MATH 132(DUE 2-25 AT START OF CLASS) PROBLEM SET II

Please work the problems in the white space provided and clearly box your solutions. If there is not enough space please write the answer in the given space and indicate that the work is supplied on a separate sheet. Be sure to label attached sheets with the Problem number so the grader can efficiently judge your solution. Enjoy!

Problem 17 Suppose C is the set of all points $(x, y) \in \mathbb{R}^2$ which solve $xy - y = 1$. The goal of this problem is simply to examine how C can be described as a level-set, graph or parametrized curve:

(a) find $F : \mathbb{R}^2 \to \mathbb{R}$ such that $F^{-1}(\{0\})$ is C

Let $F(x,y) = xy - y - 1$ then $F(x,y) = 0$ describes C and thus $F^{-1}(\{0\}) = C$.

(b) find $f : dom(f) \subseteq \mathbb{R} \to \mathbb{R}$ such that $graph(f) = C$

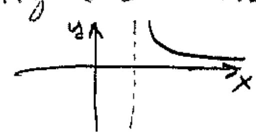
Let $f(x) = \dfrac{1}{x-1}$ for $x \neq 1$ $\left(dom(f) = \mathbb{R} - \{1\}\right)$

$$graph(f) = \left\{ (x,y) \;\middle|\; y = \frac{1}{x-1},\ x \neq 1 \right\}$$
$$= \{ (x,y) \mid xy - y = 1,\ x \neq 1 \} = C.$$

(c) find $x : J_1 \subseteq \mathbb{R} \to \mathbb{R}$ and $y : J_1 \subseteq \mathbb{R} \to \mathbb{R}$ such that $\{(x(t), y(t)) \mid t \in J_1\} \subset C$ (choose parametric equations which cover the right-branch of C)

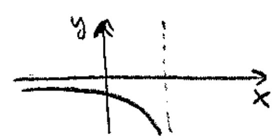
Let $x(t) = t$ and $y(t) = \dfrac{1}{t-1}$ for all $t \in J_1 = (1, \infty)$.

Notice $t \in (1, \infty) \Rightarrow 1 < t < \infty \Rightarrow$ Right Branch.



(d) find $x : J_2 \subseteq \mathbb{R} \to \mathbb{R}$ and $y : J_2 \subseteq \mathbb{R} \to \mathbb{R}$ such that $\{(x(t), y(t)) \mid t \in J_2\} \subset C$ (choose parametric equations which cover the left-branch of C)

Let $x(t) = t$ and $y(t) = \dfrac{1}{t-1}$ for all $t \in J_2 = (-\infty, 1)$. Notice this domain makes $t < 1$ hence $x < 1$ which $\Rightarrow$ Left Branch.



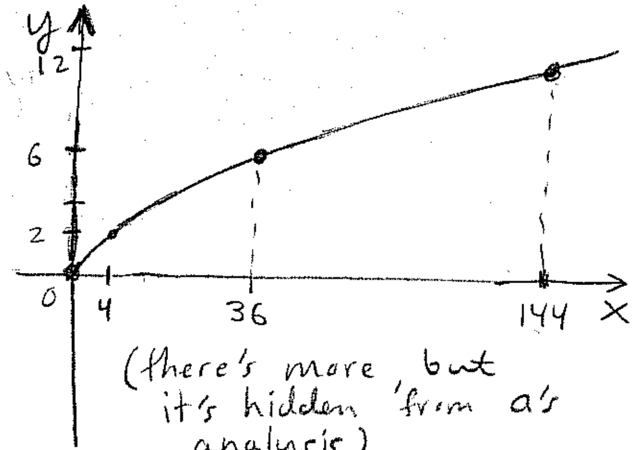
(e) use part b to write $\vec{r} : J_3 \to \mathbb{R}^2$ which covers all of C.

Let $J_3 = (-\infty, 1) \cup (1, \infty)$ and let $\vec{r}(t) = \left\langle t, \dfrac{1}{t-1} \right\rangle$ for all $t \in J_3$.

Problem 18 Suppose $x = t^4 - 2t^3 + t^2$ and $y = t^2 - t$ for $t \in [-2, 4]$ is a parametrization of the curve C.

(a) make a table of values for $t = -2, -1, 0, 1, 2, 3, 4$ and plot the curve by plotting these points connecting the dots as smoothly as possible.

	$x = t^2(t-1)^2$	$y = t(t-1)$
-2	36	6
-1	4	2
0	0	0
1	0	0
2	4	2
3	36	6
4	144	12



(there's more, but it's hidden from a's analysis)

Notice $x = \underbrace{t^2(t-1)^2}_{\text{nonnegative}}$ and $\underbrace{y = t(t-1)}_{\text{could be negative.}}$ $\left[+ + + + \mid - - - \mid + + + + \right]$ $-2 \quad 0 \quad 4$

(b) find the Cartesian form of the curve C.

$$x = t^4 - 2t^3 + t^2$$
$$= t^2(t^2 - 2t + 1)$$
$$= t^2(t-1)^2 = [t(t-1)]^2 = [t^2 - t]^2 = y^2 \quad \therefore \boxed{x = y^2}$$

(and we should add inequalities to pick out just C alone, (for $-1/4 \leq y \leq 12$ from d.)

(c) find $\vec{r} : [-2, 4] \to \mathbb{R}^2$ such that $\vec{r}([-2, 4]) = C$.

$$\vec{r}(t) = \langle t^4 - 2t^3 + t^2, \ t^2 - t \rangle$$ for each $t \in [-2, 4]$.

(d) calculate $\frac{d\vec{r}}{dt}$ and find any critical points for the path.

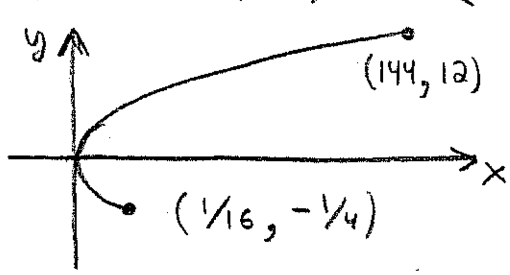
$$\frac{d\vec{r}}{dt} = \langle 4t^3 - 6t^2 + 2t, \ \underbrace{2t - 1}_{} \rangle$$

$\hookrightarrow t = 1/2$ only possible zero for dy/dt.

$$\left.\frac{dx}{dt}\right|_{t=1/2} = 4\left(\tfrac{1}{8}\right) - 6\left(\tfrac{1}{4}\right) + 2\left(\tfrac{1}{2}\right) = \tfrac{1}{2} - \tfrac{3}{2} + 1 = 0.$$

Therefore $\vec{r}(1/2) = \langle \frac{1}{4}\left(\frac{1}{4}\right), \frac{1}{2}\left(-\frac{1}{2}\right) \rangle = \langle 1/16, -1/4 \rangle$

only critical point.



Problem 19 Creating parametric equations for given level curves is generally an art. However, the questions that follow are either in my notes or are really close to something I do show: in each case find parametric equations and state the domain for the parameter which will reproduce the given curve including any constraints, you are free to use either vector or scalar notation (you can just state the answers on this page, I don't expect derivations here)

(a) find the parametrization of the linear path from $(0,1)$ to $(4,-3)$.

$$\vec{r}_a(t) = (0,1) + t\left[(4,-3)-(0,1)\right]$$
$$= \langle 0,1\rangle + t\langle 4,-4\rangle = \boxed{\langle 4t, 1-4t\rangle = \vec{r}(t) \text{ for } 0 \le t \le 1}$$

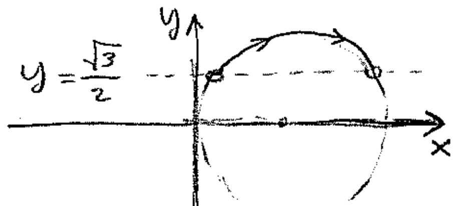
(b) find the parametrization of the linear path from $(4,-3)$ to $(0,1)$.

(other answers possible)

$\vec{r}_b(t) = \vec{r}_a(-t) = \langle -4t, 1+4t\rangle$ for $-1 \le t \le 0$

OR, $x(\tilde{t}) = 4-4\tilde{t}$, $y(\tilde{t}) = -3+4\tilde{t}$ for $0 \le \tilde{t} \le 1$

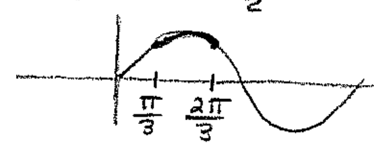
(c) find the parametric equations for $x^2-2x+y^2=0$ for $y>\sqrt{3}/2$ where the curve is oriented in the CCW-direction.



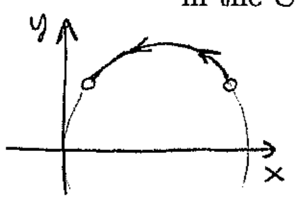
$(x-1)^2 + y^2 = 1$

Let $\boxed{x = 1+\cos t,\ y = \sin t}$ $\boxed{\pi/3 < t < 2\pi/3}$

need $\sin t > \frac{\sqrt{3}}{2}$

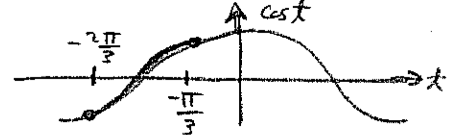


(d) find the parametric equations for $x^2-2x+y^2=0$ for $y>\sqrt{3}/2$ where the curve is oriented in the CW-direction.



$$\boxed{\begin{aligned} x(t) &= 1+\cos(t) \\ y(t) &= \sin(-t) = -\sin t \\ t &\in [-2\pi/3, -\pi/3] \end{aligned}}$$

Note $-\frac{2\pi}{3} < t < \frac{-\pi}{3}$



(e) find parametric equations for the ~~upper~~ right branch of $x^2-2(y-2)^2=1$.

$\boxed{\text{Let } x = \cosh t \text{ and } y = 2 + \tfrac{1}{2}\sinh t \text{ for } t \in \mathbb{R}}$

Note $\cosh t \ge 1$ thus $x \ge 1$.

(f) find the parametric equations which locally cover for $y^2x^2=1$ in a neighborhood of $(-1,-1)$. (Try using x as a parameter)

$$y^2 = \frac{1}{x^2} \Rightarrow y = \frac{-1}{\sqrt{x^2}} = \frac{1}{x}$$

(since $x<0 \Rightarrow \sqrt{x^2} = -x$)

$$\boxed{\vec{r}(x) = \left\langle x, \frac{1}{x}\right\rangle \text{ for } x < 0}$$

(Note $y = \frac{-1}{|x|} \Rightarrow y^2 = \frac{1}{|x|^2} = \frac{1}{x^2} \Rightarrow y^2x^2 = 1$)

(Also $\vec{r}(-1) = \langle -1,-1\rangle$ as desired)

Problem 20 Vectors have both magnitude and a direction. This direction is conveniently described by the so-called standard angle. For this problem, make your standard angles fall in the range $\theta \in [0, 2\pi)$. You'll need to do some thinking about trigonometry here, the calculator might let you down. Draw pictures to be safe. (Sorry I meant to say find θ)

(a) $\vec{A} = \langle 1, 3 \rangle$

$A = \sqrt{10} \approx 3.16$ $\qquad \tan^{-1}\left(\frac{3}{1}\right) \cong 1.249 \rightarrow \boxed{\theta_a \cong 1.249}$

(b) $\vec{B} = \langle 2, -3 \rangle$

$B = \sqrt{13} \approx 3.61$ $\qquad \tan^{-1}\left(\frac{-3}{2}\right) \cong -0.9828 \rightarrow \boxed{\theta_b \cong 5.3}$

(c) $\vec{C} = \langle -3, 0 \rangle$

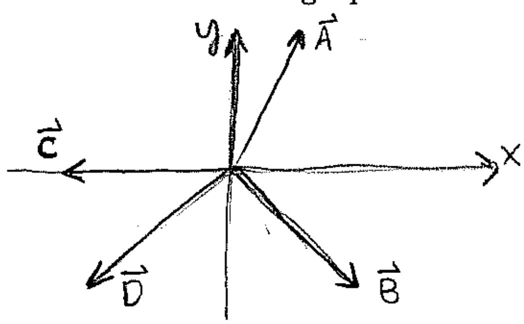
$C = 3$ $\qquad \tan^{-1}\left(\frac{0}{-3}\right) = 0 \rightarrow \boxed{\theta_c = \pi}$

(d) $\vec{D} = \langle -2, -2 \rangle$

$D = \sqrt{8} \approx 2.83$ $\qquad \tan^{-1}\left(\frac{-2}{-2}\right) = \frac{\pi}{4} \rightarrow \boxed{\theta_d = 5\pi/4}$

radians!

(e) plot all four vectors in a single plot.



Problem 21 Vectors correspond to points in a very natural manner. For each of the points below find the distance from the origin to the point and the standard **polar** angle of the given point:

(a) $A = (1, 3)$

$d_A = \sqrt{10}$, $\theta_a \cong 1.249$

(b) $B = (2, -3)$

$d_B = \sqrt{13}$, $\theta_B \cong 5.3$

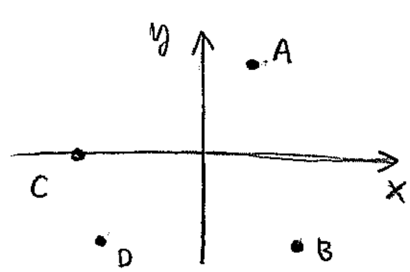
(c) $C = (-3, 0)$

$d_c = 3$, $\theta_c = \pi$

(d) $D = (-2, -2)$

$d_D = \sqrt{8}$, $\theta_D = 5\pi/4$

(e) plot all four points in a single plot.



Problem 22 Suppose $\vec{r}(t) = \langle\, t\sin(t),\ t\sin(t)\,\rangle$ denotes the position of a particle at time t.

(a) find the velocity at time t

$$\vec{V}(t) = \frac{d\vec{r}}{dt} = \boxed{\langle \sin t + t\cos t,\ \sin t + t\cos t \rangle}.$$

(b) find the acceleration at time t

$$\vec{a}(t) = \frac{d\vec{V}}{dt} = \langle \cos t + \cos t - t\sin t,\ \cos t + \cos t - t\sin t \rangle$$

$$= \boxed{\langle 2\cos t - t\sin t,\ 2\cos t - t\sin t \rangle}.$$

(c) find the displacement vector from $t = 0$ to $t = 2\pi$

$$\Delta\vec{r} = \vec{r}(2\pi) - \vec{r}(0) = \vec{0} - \vec{0} = \boxed{\vec{0}.} = \underline{\langle 0, 0 \rangle}.$$

(d) find the distance travelled from $t = 0$ to $t = 2\pi$

$$s = \int_0^{2\pi} v\, dt = \int_0^{2\pi} \sqrt{(\sin t + t\cos t)^2 + (\sin t + t\cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{2}\,|\sin t + t\cos t|\, dt$$

$$= \boxed{18.7643}$$ ← Mathematica.
(use NIntegrate)
(This kills Wolfram Alpha)

Problem 23 Suppose $\vec{a}(t) = \langle 0, -g\rangle$ is the acceleration of a particle by gravity near the surface of the earth. Futhermore, suppose the particle begins at the origin with velocity $\vec{v}(0) = \langle v_{ox}, v_{oy}\rangle$. Show that the Cartesian form of the path is a parabola. Your answer will involve the initial velocity components and g.

We can derive from twice integrating

$$\vec{a} = \frac{d\vec{v}}{dt} = \langle 0, -g\rangle \Rightarrow \vec{V}(t) = \vec{V}_o + \langle 0, -gt\rangle$$

$$\vec{V} = \frac{d\vec{r}}{dt} = \langle V_{ox}, V_{oy} - gt\rangle \Rightarrow \vec{r}(t) = \vec{r}_o + \langle V_{ox}$$

$$\Rightarrow \vec{r}(t) = \langle x_o + V_{ox}t, \; y_o + V_{oy}t - \tfrac{1}{2}gt^2\rangle$$

where I denoted $\vec{V}_o = \langle V_{ox}, V_{oy}\rangle$ and $\vec{r}_o = \langle x_o, y_o\rangle$. We have parametric eq^{n}'s

$$x = x_o + V_{ox}t \longrightarrow t = \frac{x - x_o}{V_{ox}}$$

$$y = y_o + V_{oy}t - \tfrac{1}{2}gt^2$$

$$\boxed{y = y_o + V_{oy}\left(\frac{x - x_o}{V_{ox}}\right) - \frac{1}{2}g\left(\frac{x - x_o}{V_{ox}}\right)^2}$$

If we let $x_o = 0$ then the eq^{n} simplifies w/o much trouble,

$$y = y_o + \frac{V_{oy}x}{V_{ox}} - \frac{1}{2}\frac{gx^2}{V_{ox}^2}$$

$$\Rightarrow y = y_o - \frac{g}{2V_{ox}^2}\left[x^2 - \left(\frac{V_{oy}}{V_{ox}}\right)\left(\frac{2V_{ox}^2}{g}\right)x\right]$$

$$\Rightarrow y = y_o - \frac{g}{2V_{ox}^2}\left[\left(x - \frac{V_{ox}V_{oy}}{g}\right)^2 - \left(\frac{V_{ox}V_{oy}}{g}\right)^2\right]$$

$$\Rightarrow \boxed{y = \frac{-g}{2V_{ox}^2}\left(x - \frac{V_{ox}V_{oy}}{g}\right)^2 + y_o + \frac{V_{oy}^2}{2g}}$$

Remark: Vertex at $\left(\frac{V_{ox}V_{oy}}{g}, \; y_o + \frac{V_{oy}^2}{2g}\right) = \left(\frac{V_o^2 \sin\theta\cos\theta}{g}, \; y_o + \frac{V_o^2\sin^2\theta}{2g}\right)$

where we used $V_{ox} = V_o\cos\theta$ and $V_{oy} = V_o\sin\theta$

Problem 24 Suppose $\vec{a}(t) = \langle 2, 5\rangle$ is the acceleration and $\vec{v}(1) = \langle 3, -2\rangle$ and $\vec{r}(0) = \langle 1, 1\rangle$. Find:

(a) $\vec{v}(t)$ (the velocity vector at time t)

$$\vec{a} = \langle 2, 5 \rangle = \frac{d\vec{V}}{dt} \rightsquigarrow \vec{V}(t) = \langle 2t, 5t \rangle + \vec{c}_1$$

Note, $\vec{V}(1) = \langle 3, -2 \rangle = \langle 2, 5 \rangle + \vec{c}_1$

$$\therefore \quad \vec{c}_1 = \langle 3, -2 \rangle - \langle 2, 5 \rangle = \langle 1, -7 \rangle$$

$$\Rightarrow \boxed{\vec{V}(t) = \langle 2t + 1, 5t - 7 \rangle}$$

(b) $\vec{r}(t)$ (the position vector at time t)

$$\vec{V} = \langle 2t + 1, 5t - 7 \rangle = \frac{d\vec{r}}{dt}$$

$$\vec{r}(t) = \langle t^2 + t, \tfrac{5}{2}t^2 - 7t \rangle + \vec{C}_2$$

$$\therefore \quad \vec{r}(0) = \langle 1, 1 \rangle = \langle 0, 0 \rangle + \vec{C}_2$$

$$\Rightarrow \boxed{\vec{r}(t) = \langle t^2 + t + 1, \tfrac{5}{2}t^2 - 7t + 1 \rangle}$$

(c) $s(t)$ where we take $t = 0$ as our base point. This is the arclength of the path from $t = 0$ to time $t > 0$, in other words $s(t)$ is the distance travelled during $[0, t]$.

$$s(t) = \int_0^t V(\tau)\,d\tau = \int_0^t \sqrt{(2\tau+1)^2 + (5\tau - 7)^2}\,d\tau$$

← full-credit for this, more get bonus

$$= \int_0^t \sqrt{4\tau^2 + 4\tau + 1 + 25\tau^2 - 70\tau + 49}\,d\tau$$

$$= \int_0^t \sqrt{29\tau^2 - 64\tau + 50}\,d\tau$$

$$= \boxed{\frac{1}{58}\left(\sqrt{29t^2 - 64t + 50}\right)(29t - 32) + \frac{213}{29\sqrt{29}} \sinh^{-1}\left[\frac{29t - 32}{\sqrt{426}}\right] + \frac{32}{58}\sqrt{50} + \frac{213}{29\sqrt{29}} \sinh^{-1}\left[\frac{32}{\sqrt{426}}\right]}$$

(Wolfram Alpha ⊕ FTC)

Problem 25 Suppose $\vec{r}(t) = \langle\ 2\sin(3t),\ -3\sin(2t)\ \rangle$. Find all:

(*a*.) horizontal tangents, (*b*.) vertical tangents, (*c*.) critical points

Then (d.) plot the curve (use CAS or draw neatly by hand).

a.) $\frac{dy}{dt} = -6\cos(2t) = 0$ for horizontal tangents.

$2t = \pm\pi/2,\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2},\ \pm\frac{7\pi}{2},\ \pm\frac{9\pi}{2},\ \pm\frac{11\pi}{2}, \dots$

$\underline{t = \pm\pi/4,\ \pm 3\pi/4,\ \pm 5\pi/4,\ \pm\frac{7\pi}{4},\ \pm\frac{9\pi}{4},\ \pm\frac{11\pi}{4}, \dots}$.

b.) $\frac{dx}{dt} = 6\cos(3t) = 0$ for vertical tangents

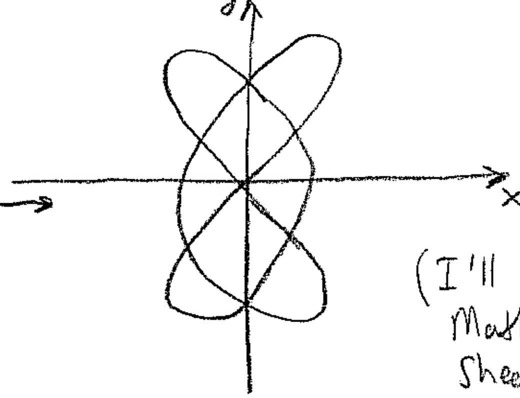
$3t = \pm\pi/2,\ \pm 3\pi/2,\ \pm 5\pi/2,\ \pm 7\pi/2,\ \pm 9\pi/2,\ \pm 11\pi/2$

$\underline{t = \pm\pi/6,\ \pm 3\pi/6,\ \pm 5\pi/6,\ \pm 7\pi/6,\ \pm 9\pi/6,\ \pm 11\pi/6, \dots}$

c.) We need $\frac{dx}{dt} = 0$ AND $\frac{dy}{dt} = 0$ for critical point since both derivatives exist $\forall t$. We should look for the lists above overlapping. If $\exists$ common sol$^{\underline{n}}$'s then those points are critical pts. For this problem $\nexists$ any critical points since the lists from a, b are distinct.

d.) (I hope you have a neater graph than this)

Parametric Plot works good for $\longrightarrow$



Remark: intersection points need not be critical!

(I'll post my Mathematica Sheet, see it for nicer pic.)

Problem 26 On parametric calculus:

(a) Problem 40 from Section 11.2 of Stewart.

(b) Problem 65 from Section 11.2 of Stewart.

(c) Problem 69 from Section 11.2 of Stewart.

(d) Problem 74 from Section 11.2 of Stewart.

(a.) Let $x = \ln(t)$ and $y = \sqrt{t+1}$ for $1 \le t \le 5$. Find arclength using some numerical method.

Note $\frac{dx}{dt} = 1/t$ and $\frac{dy}{dt} = \frac{1}{2\sqrt{t+1}}$

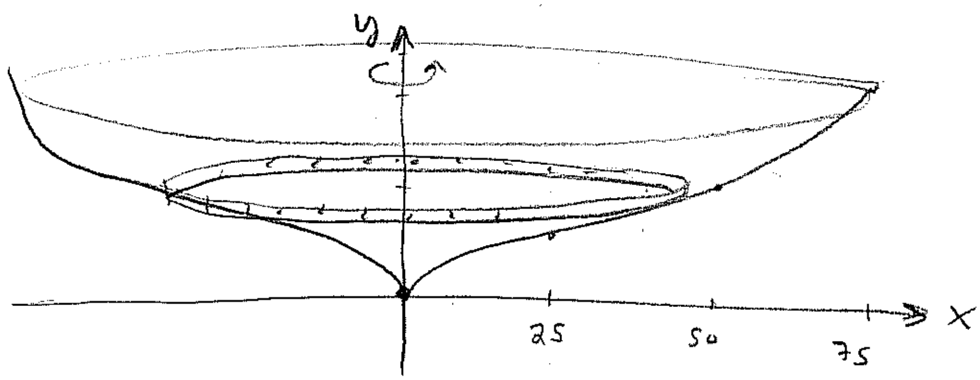
$$S = \int_1^5 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_1^5 \sqrt{\frac{1}{t^2} + \frac{1}{4(t+1)}}\, dt = \boxed{1.93101}$$

(Mathematica NIntegrate)

(b.) Let $x = 3t^2$, $y = 2t^3$ for $0 \le t \le 5$ find surface area from rotating this around y-axis.

Note $t^3 = y/2 \Rightarrow t^{1/3} = \sqrt[3]{y/2} \Rightarrow x = 3\left(\sqrt[3]{y/2}\right)^2$

t	x	y
0	0	0
1	3	2
2	12	16
3	27	54
4	48	128
5	75	250



$$dA = 2\pi r\, ds = 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad , \quad \frac{dx}{dt} = 6t, \quad \frac{dy}{dt} = 6t^2$$

$$A = \int_0^5 2\pi(3t^2)\sqrt{36t^2 + 36t^4}\, dt$$

$$= \int_0^5 36\pi t^2 \sqrt{t^2 + t^4}\, dt$$

$$= \int_0^5 36\pi t^3 \sqrt{1 + t^2}\, dt = \int_1^{26} 18\pi (u-1)\sqrt{u}\, du$$

$u = 1 + t^2$

$du = 2t\,dt$

$$= \boxed{\frac{24\pi}{5}\left(949\sqrt{26} + 1\right)}$$

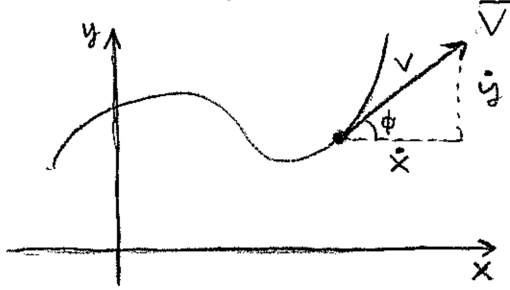
Problem 26 continued

(c.) §11.2#69 The curvature $\kappa \overset{\text{def.}}{=} \left|\frac{d\phi}{ds}\right|$ where ϕ is the angle of inclination for the tangent line and s is arclength. This gives some measure of the change in direction of a curve.

a.) Show $\kappa = \dfrac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$ for $x = x(t)$, $y = y(t)$ where $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$

b.) Show $\kappa = \dfrac{|y''|}{[1 + (y')^2]^{3/2}}$

(a.) $\vec{r}(t) = \langle x(t), y(t) \rangle$ has tangent vector $\vec{V} = \langle \dot{x}, \dot{y} \rangle$. We can picture this as follows,



$$\dot{x} = v\cos\phi \qquad v = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\dot{y} = v\sin\phi$$

$$\tan\phi = \frac{\dot{y}}{\dot{x}}$$

only good for $\dot{x} > 0$

$$\phi = \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right)$$

$$\frac{d\phi}{dt} = \frac{1}{1 + (\dot{y}/\dot{x})^2}\frac{d}{dt}\left[\frac{\dot{y}}{\dot{x}}\right]$$

$$= \left(\frac{1}{1 + \dot{y}^2/\dot{x}^2}\right)\left(\frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2}\right)$$

$$= \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2} \quad \left(\text{positive from geometry}\right)$$

Note that $\dfrac{d\phi}{dt} = \dfrac{ds}{dt}\dfrac{d\phi}{ds} \Rightarrow \underline{\dfrac{d\phi}{ds} = \dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}}$.

Geometrically, if $\dot{x} < 0$ then $-\dot{x} > 0$ and the calculation goes through for the curve $\vec{r}(t) = \langle -x, y \rangle$ and we find

$$\frac{d\phi}{ds} = \frac{-\dot{x}\ddot{y} + \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = -\left(\frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}\right) = \underbrace{\left|\frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}\right|}_{\text{negative slope from geometry}}$$

Thus $\kappa = \dfrac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$ holds as claimed.

(b.) Note, $\dot{x} = \frac{dx}{dx} = 1$ and $\ddot{x} = 0$ and it follows.

Problem 27 On arclength:

(a) $y = \ln(\sec(x))$ for $0 \le x \le \pi/4$.

(b) the path with parametrization $\vec{r}(\lambda) = \langle \lambda^2, \lambda^2 \rangle$ for $\lambda \in [0,2]$.

(c) the path with parametrization $x = 2^t\cos(t)$ and $y = 2^t\sin(t)$ for $t \in [0, \pi/2]$

(d) Problem 38 from Section 9.1 of Stewart.

d. Cow silo nobody got this?

(a.) Observe $\dfrac{dy}{dx} = \dfrac{\sec x \tan x}{\sec x} = \tan(x)$

$$s = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx$$

note $\sec(x) > 0$ for $x \in [0, \pi/4]$.

$$= \int_0^{\pi/4} \sec(x)\, dx$$

$$= \ln|\sec(x) + \tan(x)| \Big|_0^{\pi/4}$$

$$= \ln|\sqrt{2}+1| - \ln|1|$$

$$= \boxed{\ln(\sqrt{2}+1)}$$

(b.) $\vec{r}(\lambda) = \langle \lambda^2, \lambda^2 \rangle \Rightarrow \dfrac{d\vec{r}}{d\lambda} = \langle 2\lambda, 2\lambda \rangle = \vec{v}(\lambda)$

Note that $v(\lambda) = \sqrt{(2\lambda)^2 + (2\lambda)^2} = \lambda\sqrt{8}$ for $0 \le \lambda \le 2$.

$$s = \int_0^2 v(\lambda)\, d\lambda = \int_0^2 \lambda\sqrt{8}\, d\lambda = \frac{1}{2}\lambda^2\sqrt{8}\Big|_0^2 = \boxed{2\sqrt{8}}$$

Note: this is just a nonlinear parametrization of the line-segment from $(0,0)$ to $(4,4)$. We should not be surprised to find $\sqrt{16+16} = \sqrt{32} = 2\sqrt{8}$.

(c.)
$$x = 2^t \cos t \rightsquigarrow \dot{x} = \ln(2)2^t\cos t - 2^t \sin t = 2^t(\ln(2)\cos t - \sin t)$$
$$y = 2^t \sin t \rightsquigarrow \dot{y} = \ln(2)2^t \sin t + 2^t \cos t = 2^t(\ln(2)\sin t + \cos t)$$

$$\dot{x}^2 + \dot{y}^2 = (2^t)^2\left[(\ln(2)\cos t - \sin t)^2 + (\ln(2)\sin t + \cos t)^2\right]$$

$$= (2^t)^2\left[(\ln(2))^2\cos^2 t - \cancel{2\ln(2)\cos t \sin t} + \sin^2 t + (\ln(2))^2 \sin^2 t + \cancel{2\ln(2)\sin t \cos t} + \cos^2 t\right]$$

$$s = \int_0^{\pi/2} 2^t \sqrt{\ln(2)^2 + 1}\, dt = \boxed{\frac{\sqrt{\ln(2)^2+1}}{\ln(2)}\left[2^{\pi/2} - 1\right]}$$

Problem 28 On surface area:

(a) Problem 8 from Section 9.2 of Stewart.

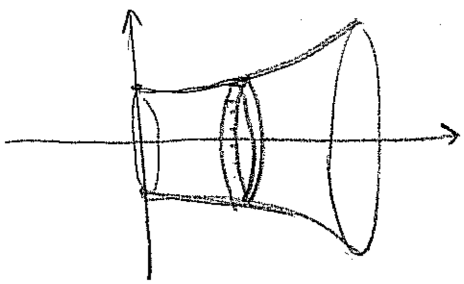
(b) Problem 16 from Section 9.2 of Stewart.

(c) Problem 28 from Section 9.2 of Stewart.

(#(c)

haven't done yet)

(a.) rotate $y = c + a\cosh(x/a)$ about x-axis, given $0 \le x \le a$.



$$dA = 2\pi r\,ds$$
$$= 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= 2\pi y\sqrt{1+\sinh^2(x/a)}\,dx$$
$$= 2\pi\left(c+a\cosh\left(\tfrac{x}{a}\right)\right)\sqrt{\cosh^2(x/a)}\,dx$$
$$= 2\pi\left[c+a\cosh\left(\tfrac{x}{a}\right)\right]\cosh\left(\tfrac{x}{a}\right)dx$$

$$A = \int_0^a \left[(2\pi c)\cosh\left(\tfrac{x}{a}\right) + 2\pi a\cosh^2\left(\tfrac{x}{a}\right)\right]dx$$
$$= (2\pi c)\,a\sinh\left(\tfrac{x}{a}\right)\Big|_0^a + \pi a\int_0^a\left[1+\cosh\left(\tfrac{2x}{a}\right)\right]dx$$
$$= (2\pi ac)\sinh(1) + \pi a\left[a + \tfrac{a}{2}\sinh\left(\tfrac{2x}{a}\right)\Big|_0^a\right]$$
$$= \boxed{2\pi ac\sinh(1) + \pi a^2 + \tfrac{1}{2}\pi a^2\sinh(2)}$$

Problem 28 part b §9.2 #16

Rotate $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$ for $1 \le x \le 2$ around y-axis,
Find surface area of the surface of revolution

$$\frac{dy}{dx} = \frac{1}{2}x - \frac{1}{2x} = \frac{1}{2x}(x^2 - 1)$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4x^2}(x^2-1)^2 = 1 + \frac{1}{4x^2}(x^4 - 2x^2 + 1)$$

$$= 1 + \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2}$$

$$= \frac{1}{4}\left(x^2 + 2 + \frac{1}{x^2}\right)$$

$$= \frac{1}{4}\left(x + \frac{1}{x}\right)^2$$

$$A = \int_1^2 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 2\pi x \sqrt{\frac{1}{4}\left(x + \frac{1}{x}\right)^2}\, dx$$

$$= \int_1^2 \pi x \left(x + \frac{1}{x}\right) dx$$

$$= \frac{\pi x^3}{3}\Big|_1^2 + \pi x\Big|_1^2$$

$$= \pi\left(\frac{8}{3} - \frac{1}{3} + 2 - 1\right)$$

$$= \pi\left(\frac{8}{3} + \frac{2}{3}\right) = \boxed{\frac{10\pi}{3}}$$

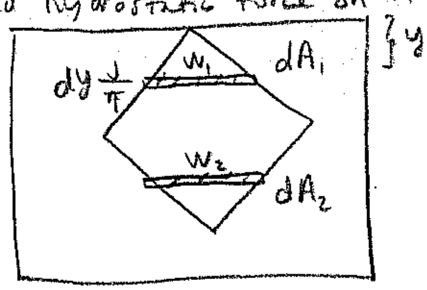
Problem 29 On physics:

(a) Problem 10 from Section 9.3 of Stewart.

(b) Problem 30 from Section 9.3 of Stewart.

(c) Problem 23 from Section 6.4 of Stewart.

(a.) Find hydrostatic force on the diamond, assume water up to top.



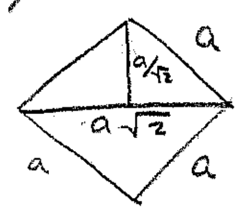
$$P(y) = \rho g y = \frac{dF}{dA}$$

$$dF_1 = \rho g y \, dA_1 = \rho g y \, w_1 \, dy$$

$$dF_2 = \rho g y \, dA_2 = \rho g y \, w_2 \, dy$$

Need to find formulas for w_1 and w_2,

$$\left.\begin{array}{l} w_1(0) = 0 \\ w_1(a/\sqrt{2}) = a\sqrt{2} \end{array}\right\} \; w_1 = y\left(\frac{a\sqrt{2}}{a/\sqrt{2}}\right) = 2y$$



$$\left.\begin{array}{l} w_2(a/\sqrt{2}) = a\sqrt{2} \\ w_2(a\sqrt{2}) = 0 \end{array}\right\}$$

Let $w_2 = m_2 y + b_2$

$$w_2(a/\sqrt{2}) = m_2(a/\sqrt{2}) + b_2 = a\sqrt{2}$$

$$w_2(a\sqrt{2}) = m_2 a\sqrt{2} + b_2 = 0$$

$$m_2\left(a\left(\tfrac{1}{\sqrt{2}} - \sqrt{2}\right)\right) = a\sqrt{2}$$

$$m_2 = \frac{\sqrt{2}}{\frac{1}{\sqrt{2}} - \sqrt{2}} = \frac{2}{1-2} = -2.$$

$$b_2 = 2a\sqrt{2}. \quad \therefore \; \underline{w_2 = -2y + 2a\sqrt{2}}$$

Add all the dF's

$$F_{net} = \int_0^{a/\sqrt{2}} \rho g y\,(2y\,dy) + \int_{a/\sqrt{2}}^{a} \rho g y\,(-2y + 2a\sqrt{2})\,dy$$

$$= \rho g \left[\frac{2y^3}{3}\Big|_0^{a/\sqrt{2}} - \frac{2y^3}{3}\Big|_{a/\sqrt{2}}^{a} + 2a\sqrt{2}\,\frac{y^2}{2}\Big|_{a/\sqrt{2}}^{a} \right]$$

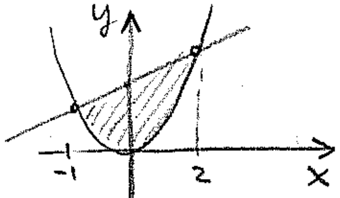
$$= \rho g \left[\frac{2}{3}\left(\frac{a}{\sqrt{2}}\right)^3 - \frac{2}{3}\left(a^3 - \left(\frac{a}{\sqrt{2}}\right)^3\right) + a\sqrt{2}\left(a^2 - \left(\frac{a}{\sqrt{2}}\right)^2\right) \right]$$

$$= \rho g \left[\frac{1}{3\sqrt{2}} - \frac{2}{3}\left(1 - \frac{1}{2\sqrt{2}}\right) + \sqrt{2}\left(1 - \frac{1}{2}\right) \right] a^2$$

$$\cong \boxed{0.5118 \rho g a^2 \cong \left(\frac{5}{6}\sqrt{2} - \frac{2}{3}\right)\rho g a^2}$$

Problem 29b §9.3 #30

Find centroid for region bounded by $y = x+2$, $y = x^2$



$x+2 = x^2$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$$\bar{y} = \frac{1}{A}\int_{-1}^{2} \frac{1}{2}\left[(x+2)^2 - x^4\right]dx$$

$$= \frac{1}{A}\int_{-1}^{2} \frac{1}{2}\left(x^2 + 4x + 4 - x^4\right)dx$$

$$= \frac{1}{A}\left[\frac{x^3}{6} + x^2 + 2x - \frac{x^5}{10}\right|_{-1}^{2} = \frac{8}{5}$$

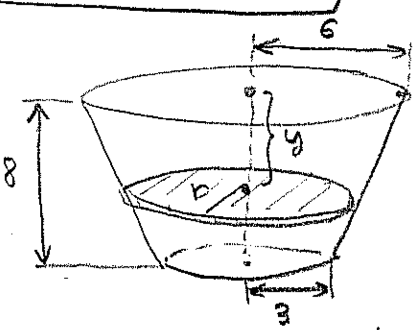
$$\bar{x} = \frac{1}{A}\int_{-1}^{2} x\left(x+2-x^2\right)dx$$

$$= \frac{1}{A}\int_{-1}^{2}\left(x^2 + 2x - x^3\right)dx$$

$$= \frac{1}{A}\left[\frac{x^3}{3} + x^2 - \frac{x^4}{4}\right|_{-1}^{2} = 1/2 \quad \therefore \underbrace{\boxed{(1/2,\ 8/5)}}_{\text{centroid}}$$

Notice $A = \int_{-1}^{2}(x+2-x^2)\,dx = \left(\frac{1}{2}x^2 + 2x - \frac{x^3}{3}\right/_{-1}^{2} = \frac{9}{2}$

Problem 29c, §6.4 #23



Note r must be linear function of y thus $r(0) = 6$ and $r(8) = 3$ completely determine the formula for $r = r(y)$,

$$r(y) = 6 - \frac{3}{8}y \quad \to r(0) = 6 \checkmark, \quad r(8) = 3 \checkmark$$

Note $dV = \pi r^2 dy$ for the disk pictured and we need to lift it a distance y against gravity.

$$dW = \left(F_{\text{gravity}} \text{ on } dV\right)y = \underbrace{\left(62.5\,\frac{lb}{ft^3}\right)}_{\text{force per unit volume.}}\underbrace{\pi r^2 y\,dy}_{\text{volume} \times \text{distance}} \Big\}\ \text{force} \cdot \text{distance}$$

$$W = \int_0^8 \pi(62.5)\left[6 - \frac{3y}{8}\right]^2 y\,dy$$

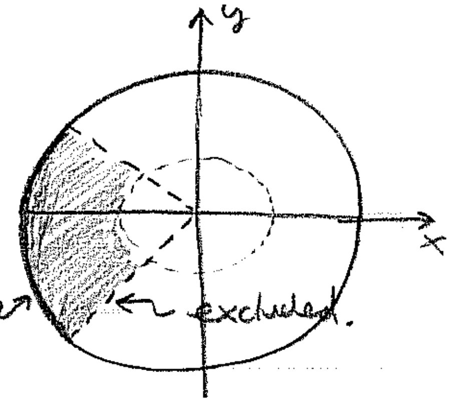
$$\cong \boxed{1.04 \times 10^5 \text{ ft-lbs}}$$

Problem 30 [worth 3pts, 2 of which are bonus!] On polar geometry:

(a) Problem 10 from Section 11.3 of Stewart.
(b) Problem 20 from Section 11.3 of Stewart.
(c) Problem 24 from Section 11.3 of Stewart.
(d) Problem 30 from Section 11.3 of Stewart.
(e) Problem 42 from Section 11.3 of Stewart.
(f) Problem 60 from Section 11.3 of Stewart.
(g) Problem 66 from Section 11.3 of Stewart.
(h) Problem 70 from Section 11.3 of Stewart.
(i) Problem 79 from Section 11.3 of Stewart.

(a.) $\underbrace{2 < r \leq 5 \text{ and } 3\pi/4 < \theta < 5\pi/4}$

Sketch this region →



(b.) $r = \tan\theta \sec\theta$

find Cartesian form of this curve.

$r = \frac{\sin\theta}{\cos^2\theta} \rightsquigarrow r\cos^2\theta = \sin\theta \rightsquigarrow r^2\cos^2\theta = r\sin\theta$

$(r\cos\theta)^2 = r\sin\theta$

$\therefore \boxed{y = x^2}$

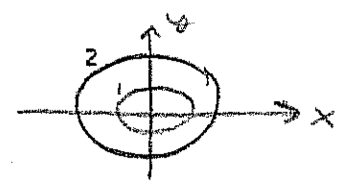
(c.) $\underbrace{x + y = 9}$

convert to a polar eq$^{\underline{n}}$.

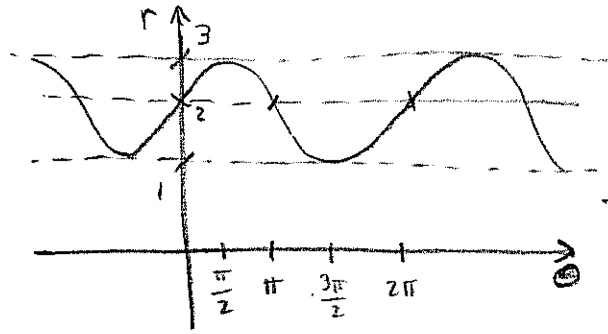
$r\cos\theta + r\sin\theta = 9 \longrightarrow \boxed{r = \frac{9}{\cos\theta + \sin\theta}}$

(d.) Sketch graph of $r^2 - 3r + 2 = 0$

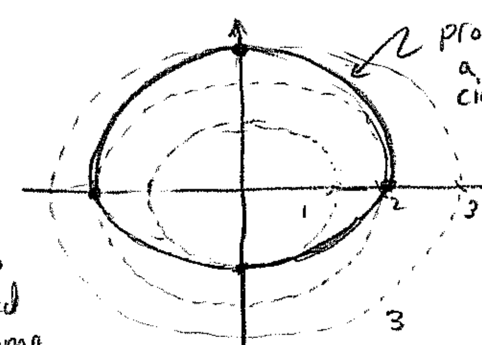
$\hookrightarrow (r-1)(r-2) = 0 \rightsquigarrow \begin{matrix} r=1 \\ r=2 \end{matrix}$



(e.) $r = 2 + \sin\theta$ Sketch this curve.



notice how the dotted lines become circular guides



probably a circle once I clean this up!

Problem 30

(f) §11.3 #60 Find slope of tangent line to $r = \cos(\theta/3)$ for $\theta = \pi$

We have $x = \cos(\theta/3)\cos\theta \rightsquigarrow \frac{dx}{d\theta} = -\frac{1}{3}\sin\left(\frac{\theta}{3}\right)\cos\theta - \cos\left(\frac{\theta}{3}\right)\sin\theta$

$y = \cos(\theta/3)\sin\theta \rightsquigarrow \frac{dy}{d\theta} = -\frac{1}{3}\sin\left(\frac{\theta}{3}\right)\sin\theta + \cos\left(\frac{\theta}{3}\right)\cos\theta$

$$\left.\frac{dx}{d\theta}\right|_{\pi} = -\frac{1}{3}\sin\left(\frac{\pi}{3}\right)\cos(\pi) = \frac{1}{3}\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{6}$$

$$\left.\frac{dy}{d\theta}\right|_{\pi} = \cos\left(\frac{\pi}{3}\right)\cos(\pi) = -\frac{1}{2}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\pi} = \frac{-1/2}{\sqrt{3}/6} = \frac{-6}{2\sqrt{3}} = \frac{-3}{\sqrt{3}} = \boxed{-\sqrt{3}}$$

(g.) §11.3 #66 Find horizontal and vert. tangents for the polar curve $r = e^\theta$

Parametrically, using θ as parameter, this gives

$x = e^\theta\cos\theta \rightsquigarrow \frac{dx}{d\theta} = 0$ for vertical tangents

$y = e^\theta\sin\theta \longrightarrow \frac{dy}{d\theta} = 0$ for horizontal tangents

didn't happen here — when both zero, need to approach via limit. (could be either or neither ...)

$$\frac{dx}{d\theta} = e^\theta(\cos\theta - \sin\theta) = 0 \longrightarrow \sin\theta = \cos\theta \longrightarrow \tan\theta = 1 \longrightarrow \boxed{\theta = \frac{\pi}{4} + \pi k \text{ for } k \in \mathbb{Z}}$$

vertical tangents.

$$\frac{dy}{d\theta} = e^\theta(\sin\theta + \cos\theta) = 0 \longrightarrow \sin\theta = -\cos\theta \longrightarrow \tan\theta = -1 \longrightarrow \boxed{\theta = \frac{-\pi}{4} + \pi k \text{ for } k \in \mathbb{Z}}$$

horizontal tangents.

Problem 30 continued

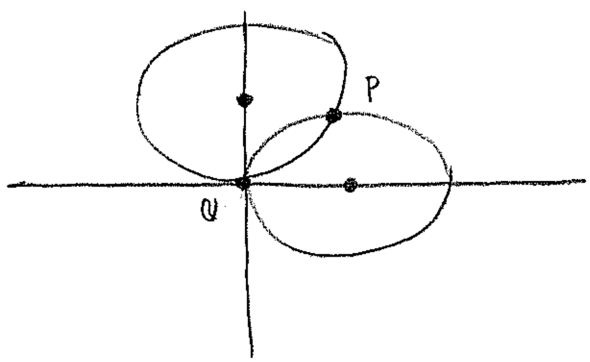
(h.) $\boxed{\S 11.3 \# 70}$ Show that $r = a\sin\theta$ and $r = a\cos\theta$ intersect at right angles

Note, $r^2 = ar\sin\theta$ and $r^2 = ar\cos\theta$

$x^2+y^2 = ay$ and $x^2+y^2 = ax$

$\underbrace{x^2 + \left(y - \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2}$ and $\underbrace{\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2}$

circle at $(0, a/2)$ of radius $a/2$ — circle at $(a/2, 0)$ of radius $a/2$



if I used a compass, it would be obvious the tangents to P and Q of these circles are $\perp$

(Give me my slice ☺)

Remark: the calculus sol^{n} is simply to calculate $\left.\frac{dy}{dx}\right|_{r=a\sin\theta}$ and $\left.\frac{dy}{dx}\right|_{r=a\cos\theta}$ to show they are negative reciprocal at points of intersection.

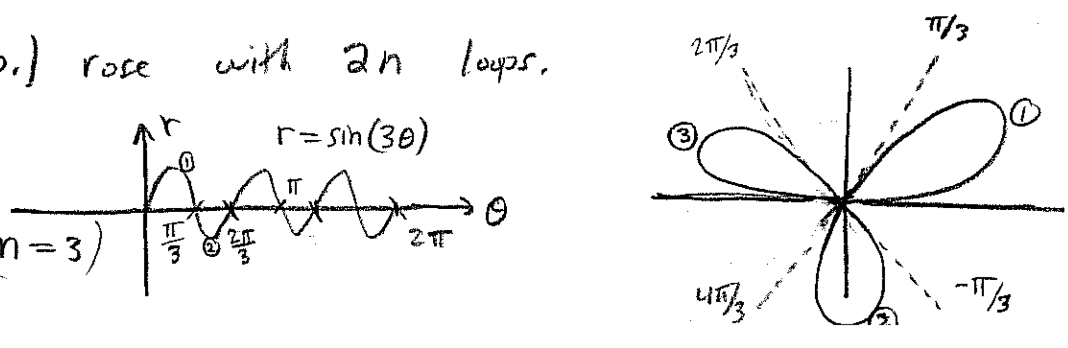
(i.) investigate $r = \sin(n\theta)$
investigate $r = |\sin(n\theta)|$ } parts a & b of $\S 11.3 \# 79$.

a.) rose with $\begin{cases} n\text{-loops} & \text{if } n \text{ odd} \\ 2n\text{-loops} & \text{if } n \text{ even} \end{cases}$

b.) rose with $2n$ loops.

(not a proof!)

$(n=3)$



if we look at $r = |\sin 3\theta|$ it adds 3 more petals

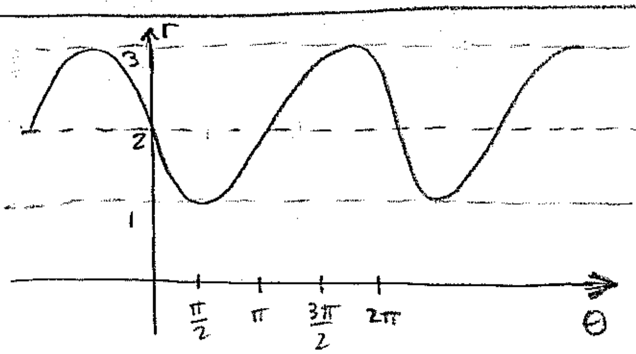
Problem 31 On polar coordinate-based calculus:

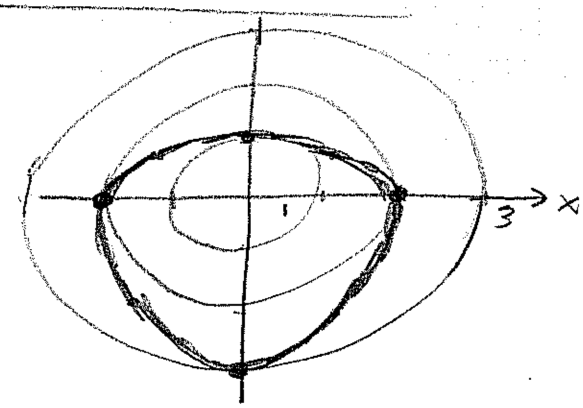
(a) Problem 12 from Section 11.4 of Stewart.

(b) Problem 18 from Section 11.4 of Stewart.

(c) Problem 26 from Section 11.4 of Stewart.

(a.) Sketch $r = 2 - \sin\theta$ and find area enclosed





It's clear this bounds a closed region around origin $0 \le r \le 2 - \sin\theta$ for $0 \le \theta \le 2\pi$

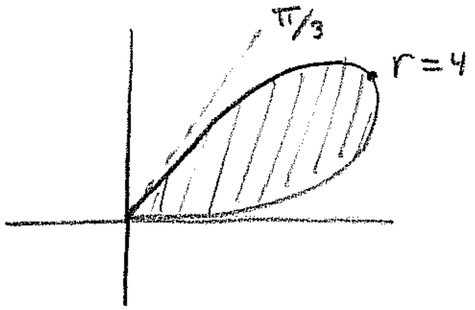
$$A = \int_0^{2\pi} \frac{1}{2}(2 - \sin\theta)^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\left(4 - \cancelto{0}{4\sin\theta} + \sin^2\theta\right) d\theta$$

$$= \int_0^{2\pi} \left[2 + \frac{1}{4}\left(1 - \cos(2\theta)\right)\right] d\theta$$

$$= \int_0^{2\pi} \left(\frac{9}{4} - \frac{1}{4}\cancelto{0}{\cos(2\theta)}\right) d\theta = \boxed{\frac{9\pi}{2}}$$

(b.) $r = 4\sin 3\theta$, calculate area inside one loop



$$A = \int_0^{\pi/3} \frac{1}{2}(4\sin 3\theta)^2 \, d\theta$$

$$= \int_0^{\pi/3} 8\sin^2 3\theta \, d\theta$$

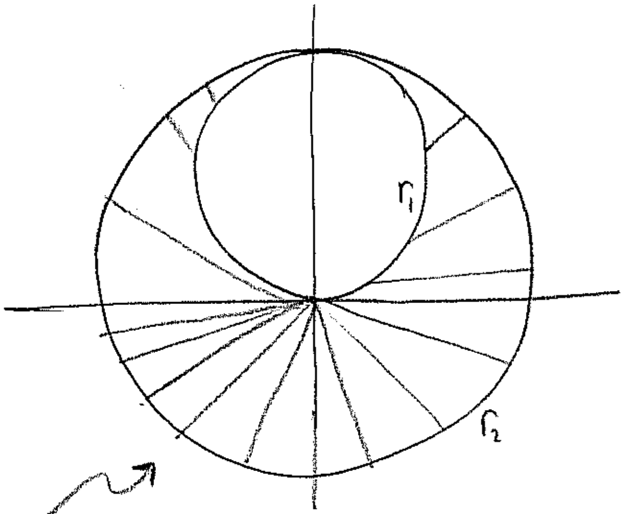
$$= \int_0^{\pi/3} 4\left(1 - \cos(6\theta)\right) d\theta$$

$$= \left[4\theta - \frac{4}{6}\sin(6\theta)\right|_0^{\pi/3}$$

$$= \frac{4\pi}{3} - \frac{4}{6}\sin\cancelto{0}{\left(\frac{6\pi}{3}\right)} = \boxed{\frac{4\pi}{3}}$$

Problem 31c | § 11.4 # 26

Find area between $r_1 = 2 + \sin\theta$ and $r_2 = 3\sin\theta$



two cases

1.) for $0 \le \theta \le \pi$

$3\sin\theta \le r \le 2 + \sin\theta$

2.) for $\pi \le \theta \le 2\pi$

$0 \le r \le 3\sin\theta$

I like to draw lines radially from origin to help me see the radial bounds. This helped me see difference between upper & lower half.

$$A = \int_0^{\pi} \frac{1}{2}\left[(2+\sin\theta)^2 - (3\sin\theta)^2\right] d\theta + \int_{\pi}^{2\pi} \frac{9\sin^2\theta}{2}\, d\theta$$

$$= \underbrace{\int_0^{\pi} \frac{1}{2}\left[4 + 4\sin\theta - 8\sin^2\theta\right] d\theta}_{4} + \int_{\pi}^{2\pi} \frac{9}{2}\sin^2\theta\, d\theta$$

$$= 4 + 9\pi/4$$

$$= \boxed{\frac{9\pi}{4} + 4}$$

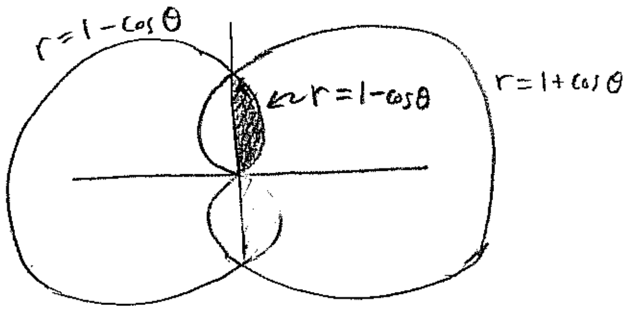
Problem 32 On polar coordinate-based calculus:

(a) Problem 30 from Section 11.4 of Stewart.

(b) Problem 38 from Section 11.4 of Stewart.

(c) Problem 46 from Section 11.4 of Stewart.

(a.) $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$ bounds some area. Find the area between both curves.



By symmetry $A = 4A_{shaded}$.

A_{shaded} described by

$0 \leq \theta \leq \pi/2$

$0 < r < 1 - \cos\theta$

$$A_{shaded} = \int_0^{\pi/2} \frac{1}{2}(1-\cos\theta)^2\, d\theta = \frac{3\pi}{8} - 1$$

$$\hookrightarrow \boxed{A = \frac{3\pi}{2} - 4 \approx 0.7124}$$

(b.) Suppose $r = 1 - \cos\theta$ and $r = 1 + \sin\theta$ find all points of intersection.

Note $r^2 = r - r\cos\theta$ and $r^2 = r + r\sin\theta$

$\hookrightarrow$ at intersection pts both hold true $\rightarrow$ $r^2 = r - r\cos\theta = r + r\sin\theta$

$\underbrace{\text{either } r = 0}_{\text{origin.}}$ or $\underbrace{-\cos\theta = \sin\theta}$

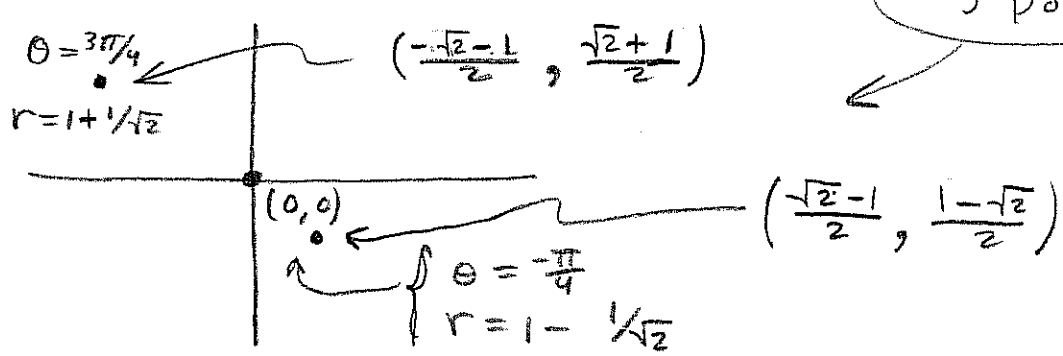
$\tan\theta = -1$

$\theta = \frac{3\pi}{4}$ or $\frac{-\pi}{4}$

If $\theta = \frac{-\pi}{4}$ then $r = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$

If $\theta = \frac{3\pi}{4}$ then $r = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$

3 points



Problem 32 c) §11.4 #46

Find the length of $r = e^{2\theta}$ for $0 \leq \theta \leq 2\pi$

$$\left.\begin{aligned} x &= e^{2\theta}\cos\theta \\ y &= e^{2\theta}\sin\theta \end{aligned}\right\}$$

use θ as parameter and subst. into
$x = r\cos\theta$
$y = r\sin\theta$
as is our custom.

$$\frac{dx}{d\theta} = e^{2\theta}(2\cos\theta - \sin\theta)$$

$$\frac{dy}{d\theta} = e^{2\theta}(2\sin\theta + \cos\theta)$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (e^{2\theta})^2\left[4\cos^2\theta - \cancel{4\sin\theta\cos\theta} + \sin^2\theta + 4\sin^2\theta + \cancel{4\sin\theta\cos\theta} + \cos^2\theta\right]$$

$$= (e^{2\theta})^2\left[4(\cos^2\theta + \sin^2\theta) + \sin^2\theta + \cos^2\theta\right]$$

$$= 5(e^{2\theta})^2$$

$$S = \int_0^{2\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= \int_0^{2\pi}\sqrt{5(e^{2\theta})^2}\,d\theta$$

$$= \sqrt{5}\int_0^{2\pi} e^{2\theta}\,d\theta$$

$$= \boxed{\frac{\sqrt{5}}{2}\left(e^{4\pi} - 1\right)}$$